Review Mechanics of OLS and Matrices: Solutions

1.(a) Use

i	x_i	y_i	$(x_i-\bar{x})^2$	$(y_i-\bar{y})^2$	$(x_i-\bar{x})(y_i-\bar{y})$
1	1	1	$-1^2=1$	$-2^2=4$	$-1\times-2=2$
2	2	3	$0^2=0$	$0^2=0$	$0\times 0=0$
3	2	4	$0^2=0$	$1^2=1$	$0\times 1=0$
4	3	4	$1^2=1$	$1^2=1$	$1\times 1=1$
Sum	8	12	2	6	3
Average	$\bar{x}=2$	$\bar{y}=3$			

So $\bar{x}=2$, $\bar{y}=3$, $\sum_{i=1}^4(x_i-\bar{x})^2=2$, $\sum_{i=1}^4(y_i-\bar{y})^2=6$, and $\sum_{i=1}^4(x_i-\bar{x})(y_i-\bar{y})=3$.

(b) $\widehat{\beta}_2=\dfrac{\sum_{i=1}^n(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^n(x_i-\bar{x})^2}=\frac{3}{2}=1.5$;

and $\widehat{\beta}_1=\bar{y}-\widehat{\beta}_2\bar{x}=3-1.5\times 2=0$.

(c) $\widehat{y}_i=0+1.5x_i$ equals 1.5, 3, 3 and 4.5 at $x=1$, 2, 2, and 3.

This yields $\sum_{i=1}^n(y_i-\widehat{y}_i)^2=(-0.5)^2+0^2+1^2+(-0.5)^2=1.5$ and hence $R^2=1-1.5/6=0.75$.

(d) $r_{xy}=\dfrac{\sum_{i=1}^N(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum_{i=1}^N(x_i-\bar{x})^2\times\sum_{i=1}^N(y_i-\bar{y})^2}}=\frac{3}{\sqrt{2\times 6}}=\frac{3}{\sqrt{12}}=0.866$.

Here $r_{xy}^2=(3/\sqrt{12})^2=9/12=0.75=R^2$ as expected.

(e) Compute $s^2=\frac{1}{n-2}\sum_{i=1}^n(y_i-\widehat{y}_i)^2=\frac{1}{2}\times 1.5=0.75$ and hence $\widehat{\mathrm{V}}[\widehat{\beta}_2]=0.75/2=0.375$.

(f) Here $t=(\widehat{\beta}_2-0)/s_{\widehat{\beta}_2}=1.5/\sqrt{0.375}=2.449$.

Since $|t|<t_{.025;2}=4.303$ we do not reject H_0 at significance level 0.05.

Alternative $p=\Pr[|t|>2.449|t\sim T_2]=2\times ttail(2,2.449)=2\times 0.067=0.134$.

2.(a)-(b) Straightforward.

(c) `summarize` yields

```
    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
           y |         4           3    1.414214          1          4
           x |         4           2    .8164966          1          3
```

so $\bar{x}=2$, $\bar{y}=3$, $\sum_{i=1}^4(x_i-\bar{x})^2=3\times s_x^2=3\times .8164966^2=2$ and $\sum_{i=1}^4(y_i-\bar{y})^2=3\times s_y^2=3\times 1.414214^2=6$.

(d) `regress` yields R-squared $=0.7500$ and

```
------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |        1.5   .6123724     2.45   0.134    -1.134826    4.134826
       _cons |  -8.88e-16   1.299038    -0.00   1.000     -5.58931     5.58931
------------------------------------------------------------------------------
```

(e) These are same as before. Also note same standard error and p-value for slope coefficient.

(f) `correlate` yields correlation between y and x of 0.8660.

3.(a)

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 18 \end{bmatrix} \qquad \text{and} \qquad \mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 4 \end{bmatrix}$$

(b)

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} 4 & 8 \\ 8 & 18 \end{bmatrix}^{-1} = \frac{1}{72-64}\begin{bmatrix} 18 & -8 \\ -8 & 4 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 18 & -8 \\ -8 & 4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 9 & -4 \\ -4 & 2 \end{bmatrix}$$

(c)

$$\mathbf{X}'\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 27 \end{bmatrix}.$$

(d)

$$(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \frac{1}{4}\begin{bmatrix} 9 & -4 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 12 \\ 27 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 108-108 \\ -48+54 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}.$$

(e) $\widehat{\mathbf{y}} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 3 \\ 3 \\ 4.5 \end{bmatrix}$. So $(\mathbf{y} - \widehat{\mathbf{y}}) = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 1.5 \\ 3 \\ 3 \\ 4.5 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0 \\ 1 \\ -0.5 \end{bmatrix}$.

Compute $s^2 = \frac{1}{2}(\mathbf{y} - \widehat{\mathbf{y}})'(\mathbf{y} - \widehat{\mathbf{y}}) = \frac{1}{2}\begin{bmatrix} -0.5 & 0 & 1 & -0.5 \end{bmatrix}\begin{bmatrix} -0.5 \\ 0 \\ 1 \\ -0.5 \end{bmatrix} = \frac{1}{2} \times 1.5 = 0.75.$

(f) Compute $\mathrm{V}[\widehat{\boldsymbol{\beta}}] = s^2(\mathbf{X}'\mathbf{X})^{-1} = 0.75 \times \frac{1}{4}\begin{bmatrix} 9 & -4 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1.6875 & -0.75 \\ -0.75 & 0.375 \end{bmatrix}$.

(g) $\bar{y} = 3$ so $(\mathbf{y} - \bar{y}\mathbf{e}) = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ and $(\mathbf{y} - \bar{y}\mathbf{e})'(\mathbf{y} - \bar{y}\mathbf{e}) = 6$.

Compute $R^2 = 1 - (\mathbf{y} - \widehat{\mathbf{y}})'(\mathbf{y} - \widehat{\mathbf{y}})/(\mathbf{y} - \bar{y}\mathbf{e})'(\mathbf{y} - \bar{y}\mathbf{e}) = 1 - 1.5/6 = 0.75$.

(h) Results are the same. Note that the square root of the diagonal entries of $s^2(\mathbf{X}'\mathbf{X})^{-1}$ are $\sqrt{1.6875} = 1.299038$ and $\sqrt{0.375} = 1.7875$, which are the standard errors given in the Stata output.

4.(a) (i) $\mathbf{A}$ is upper triangular as only $0's$ below the diagonal.

(ii) $\mathbf{A}$ is not symmetric as $\mathbf{A}' = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \neq \mathbf{A}$.

(iii) $\mathbf{A}$ is not orthogonal $\mathbf{A}'\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \neq \mathbf{I}_2$.

(iv) $\mathbf{A}$ is not idempotent as $\mathbf{AA} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \neq \mathbf{A}$.

(b) $|\mathbf{A}| = 1 \times 1 - 2 \times 0 = 1$

$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$. Check: $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) $\mathbf{x}'\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix} = x_1^2 + 2x_1x_2 + x_2^2 = (x_1 + x_2)^2$.

Positive semidefinite as $(x_1 + x_2)^2 \geq 0$ for all x_1 and x_2.

Not positive definite, however, as e.g. $x_1 = 1$ and $x_2 = -1$ gives $(x_1 + x_2)^2 = 0$.

(d) Check column rank by considering linear combinations of columns:

$$a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} a + b &= 0 \quad ...(1) \\ b &= 0 \quad ...(2) \end{aligned}$$

By (2) we must have $b = 0$.

But then by (1) we must have $a = 0$.

So Rank[$\mathbf{A}$] = 2, since 2 columns are linearly independent.

5.(a) $\widehat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}(\mathbf{X}\boldsymbol{\beta}+\mathbf{u}) = (\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}\mathbf{X}\boldsymbol{\beta}+(\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}\mathbf{u} = \boldsymbol{\beta}+(\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}\mathbf{u}$.

(b) $\mathrm{E}[\widehat{\boldsymbol{\beta}}] = \mathrm{E}[\boldsymbol{\beta} + (\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}\mathbf{u}] = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}\mathrm{E}[\mathbf{u}] = \boldsymbol{\beta}$ as $\mathrm{E}[\mathbf{u}] = \mathbf{0}$.

(c) $\mathrm{V}[\widehat{\boldsymbol{\beta}}] = \mathrm{E}[((\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}\mathbf{u})((\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}\mathbf{u})']$

$= \mathrm{E}[(\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}\mathbf{u}\,\mathbf{u}'\mathbf{W}\mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}] = (\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}\mathrm{E}[\mathbf{u}\,\mathbf{u}']\mathbf{W}\mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}$

$= (\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}\boldsymbol{\Omega}\mathbf{W}\mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}$.

(d) Yes as $\widehat{\boldsymbol{\beta}}$ is a linear transformation of the normally distributed $\mathbf{u}$, and linear transformations of the normal are also normally distributed.