240A Solutions to Final Exam Winter 2006

1.(a) We have $\mathbf{x}_i'\widehat{\boldsymbol{\beta}}$ as prediction of $y_i = \mathbf{x}_i'\boldsymbol{\beta} + u_i$. under classical assumptions including normal error,

$$\begin{array}{rc} & y_i - \mathbf{x}_i^{*\prime}\widehat{\boldsymbol{\beta}} = \mathbf{x}_i^{*\prime}\boldsymbol{\beta} + u_i - \mathbf{x}_i^{*\prime}\widehat{\boldsymbol{\beta}} = \mathbf{x}_i^{*\prime}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) + u_i \sim \mathcal{N}[\mathbf{0}, \mathbf{x}_i^{*\prime}\sigma^2(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_i^* + \sigma^2] \\ \Rightarrow & y_i - \mathbf{x}_i^{*\prime}\widehat{\boldsymbol{\beta}} \sim \mathcal{N}[\mathbf{0}, \sigma^2\{1 + \mathbf{x}_i^{*\prime}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_i^*\}] \\ \Rightarrow & \frac{y_i - \mathbf{x}_i^{*\prime}\widehat{\boldsymbol{\beta}}}{\sqrt{\sigma^2\{1 + \mathbf{x}_i^{*\prime}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_i^*\}}} \sim \mathcal{N}[\mathbf{0}, 1] \\ \Rightarrow & \frac{y_i - \mathbf{x}_i^{*\prime}\widehat{\boldsymbol{\beta}}}{\sqrt{s^2\{1 + \mathbf{x}_i^{*\prime}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_i^*\}}} \sim T_{N-k} \\ \Rightarrow & 95\% \text{ CI is } \boldsymbol{\beta} \in \widehat{\boldsymbol{\beta}} \pm t_{N-k;.025} \times \sqrt{s^2\{1 + \mathbf{x}_i^{*\prime}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_i^*\}}. \end{array}$$

(b) We have

$$\begin{aligned} \widehat{\boldsymbol{\beta}}_1 &= (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'[\mathbf{X}_1'\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 + \mathbf{u}] = \boldsymbol{\beta}_1 + (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2\boldsymbol{\beta}_2 + (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{u}. \\ \mathrm{E}[\widehat{\boldsymbol{\beta}}_1] &= \boldsymbol{\beta}_1 + (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2\boldsymbol{\beta}_2 \text{ as } \mathrm{E}[\mathbf{u}] = \mathbf{0}. \end{aligned}$$

Conclude that OLS is biased unless $\mathbf{X}_1'\mathbf{X}_2 = \mathbf{0}$.

(c) We have

$$\mathbf{y} = \left[\begin{array}{cc} \mathbf{l} & \mathbf{X}_2^* \end{array} \right] \left[\begin{array}{c} \alpha_1 \\ \boldsymbol{\beta}_2 \end{array} \right] + \mathbf{u} = \mathbf{Z}\boldsymbol{\gamma} + \mathbf{u}.$$

The usual $(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y}$ becomes

$$\begin{aligned} \left[\begin{array}{c} \widehat{\alpha}_1 \\ \widehat{\boldsymbol{\beta}}_2 \end{array} \right] &= \left[\begin{array}{cc} \mathbf{l}'\mathbf{l} & \mathbf{X}_2^{*\prime}\mathbf{l} \\ \mathbf{l}'\mathbf{X}_2^{*\prime} & \mathbf{X}_2^{*\prime}\mathbf{X}_2^* \end{array} \right]^{-1} \left[\begin{array}{c} \mathbf{l}'\mathbf{y} \\ \mathbf{X}_2^{*\prime}\mathbf{y} \end{array} \right] = \left[\begin{array}{cc} N & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2^{*\prime}\mathbf{X}_2^* \end{array} \right]^{-1} \left[\begin{array}{c} \mathbf{l}'\mathbf{y} \\ \mathbf{X}_2^{*\prime}\mathbf{y} \end{array} \right] \\ &= \left[\begin{array}{cc} \frac{1}{N} & \mathbf{0} \\ \mathbf{0} & (\mathbf{X}_2^{*\prime}\mathbf{X}_2^*)^{-1} \end{array} \right] \left[\begin{array}{c} \mathbf{l}'\mathbf{y} \\ \mathbf{X}_2^{*\prime}\mathbf{y} \end{array} \right] = \left[\begin{array}{c} \frac{1}{N}\mathbf{l}'\mathbf{y} \\ (\mathbf{X}_2^{*\prime}\mathbf{X}_2^*)^{-1}\mathbf{X}_2^{*\prime}\mathbf{y} \end{array} \right] = \left[\begin{array}{c} \bar{y} \\ (\mathbf{X}_2^{*\prime}\mathbf{X}_2^*)^{-1}\mathbf{X}_2^{*\prime}\mathbf{y} \end{array} \right] \end{aligned}$$

2.(a) We have $\widehat{\boldsymbol{\beta}} \sim \mathcal{N}[\boldsymbol{\beta}, \sigma^2(\mathbf{X}'\mathbf{X})^{-1}]$, so $\mathbf{R}\widehat{\boldsymbol{\beta}} - r \sim \mathcal{N}[\mathbf{R}\boldsymbol{\beta} - \mathbf{r}, \sigma^2\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}']$.
Under H_0 this simplifies as $\mathbf{R}\boldsymbol{\beta} - \mathbf{r} = \mathbf{0}$, so $\mathbf{R}\widehat{\boldsymbol{\beta}} - r \sim \mathcal{N}[\mathbf{0}, \sigma^2\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}']$.
Forming the quadratic gives the chi-square test statistic (assuming rank$[\mathbf{R}] = q$)

$$W = (\mathbf{R}\widehat{\boldsymbol{\beta}} - \mathbf{r})'[\sigma^2\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}']^{-1}(\mathbf{R}\widehat{\boldsymbol{\beta}} - \mathbf{r}) \sim \chi^2(q)$$

(b) Here $\mathbf{R} = [1 \;\; -2]$ and $r = 0$, so $\mathbf{R}\widehat{\boldsymbol{\beta}} - r = [1 \;\; -2] \left[\begin{array}{c} 5 \\ 2 \end{array} \right] = 1$, and

$$\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}' = \mathbf{R} \left[\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array} \right]^{-1} \mathbf{R}' = [1 \;\; -2] \left[\begin{array}{cc} 2 & -1 \\ -1 & 1 \end{array} \right] \left[\begin{array}{c} 1 \\ -2 \end{array} \right] = \left[\begin{array}{cc} 4 & -3 \end{array} \right] \left[\begin{array}{c} 1 \\ -2 \end{array} \right] = 10.$$

So $W = (1)'[0.1 \times 10]^{-1}(1) = 1$.
The critical value is $\chi_1^2(0.05) = z_{,025}^2 = 1.96^2 = 3.84$.
Since $W < 3.84$ we do not reject $H_0 : \beta_1 = 2\beta_2$.
[Since $q = 1$ here this can also be done as a z-test].

(c) Asymptotically we can replace σ^2 by a consistent estimate s^2 such as $s^2 = \widehat{\mathbf{u}}'\widehat{\mathbf{u}}/(N-k)$.
Then $W^* = (\mathbf{R}\widehat{\boldsymbol{\beta}} - \mathbf{r})'[s^2\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}']^{-1}(\mathbf{R}\widehat{\boldsymbol{\beta}} - \mathbf{r}) \sim \chi^2(q)$.
Alternatively can use $F = W^*/q = (\mathbf{R}\widehat{\boldsymbol{\beta}} - \mathbf{r})'[s^2\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}']^{-1}(\mathbf{R}\widehat{\boldsymbol{\beta}} - \mathbf{r})/q \sim F(q, N-k)$

3. Various topics

(a) An adequate answer is that a sequence of random variables $b_N \xrightarrow{p} b$ if for any $\varepsilon > 0$

$$\lim_{N\to\infty} \Pr[|b_N - b| < \varepsilon] = 1.$$

(b) A multivariate central limit places conditions on the vector components $\mathbf{x}_i$ of the vector average $\bar{\mathbf{X}}_N$ such that

$$\left(\mathrm{V}[\bar{\mathbf{X}}_N]\right)^{-1/2} (\bar{\mathbf{X}}_N - \mathrm{E}[\bar{\mathbf{X}}_N]) \xrightarrow{d} \mathcal{N}[\mathbf{0}, \mathbf{I}].$$

(c) The IV estimator (in the just-identified case) is $\widehat{\boldsymbol{\beta}}_{\mathrm{IV}} = (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'\mathbf{y}$ where $\mathbf{Z}$ is an $N \times k$ matrix of instruments with the property that $\operatorname{plim} N^{-1}\mathbf{Z}'\mathbf{u} = \mathbf{0}$.
It has the advantage of being consistent even if OLS is inconsistent due $\operatorname{plim} N^{-1}\mathbf{X}'\mathbf{u} \neq \mathbf{0}$.

(d) For $\mathbf{u} \sim [\mathbf{0}, \boldsymbol{\Sigma}]$, $\widehat{\boldsymbol{\beta}}_{\mathrm{GLS}} = (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{y}$.
The key property compared to OLS is that GLS is efficient (BLUE in the linear regression model].
It is also unbiased and consistent whenever GLS is unbiased.

(e) The variance of the OLS estimator is estimated by

$$\begin{aligned} \widehat{\mathrm{V}}[\widehat{\boldsymbol{\beta}}] &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\widehat{\boldsymbol{\Sigma}}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \text{ where } \widehat{\boldsymbol{\Sigma}} = \mathrm{Diag}[\widehat{u}_i^2] \\ &= \left(\textstyle\sum_i \mathbf{x}_i\mathbf{x}_i'\right)^{-1} \textstyle\sum_i \widehat{u}_i^2\mathbf{x}_i\mathbf{x}_i' \left(\textstyle\sum_i \mathbf{x}_i\mathbf{x}_i'\right)^{-1}. \end{aligned}$$

(f) We have $(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{Z}\mathbf{Z}'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) = \mathbf{y}'\mathbf{Z}\mathbf{Z}'\mathbf{y} - 2\mathbf{y}'\mathbf{Z}\mathbf{Z}'\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\beta}'\mathbf{X}'\mathbf{Z}\mathbf{Z}'\mathbf{X}\boldsymbol{\beta}$, so

$$\begin{aligned} &\Rightarrow \partial(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{Z}\mathbf{Z}'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})/\partial\boldsymbol{\beta} = -2\mathbf{X}'\mathbf{Z}\mathbf{Z}'\mathbf{y} + 2\mathbf{X}'\mathbf{Z}\mathbf{Z}'\mathbf{X}\boldsymbol{\beta} = \mathbf{0} \\ &\Rightarrow \mathbf{X}'\mathbf{Z}\mathbf{Z}'\mathbf{X}\boldsymbol{\beta} = \mathbf{X}'\mathbf{Z}\mathbf{Z}'\mathbf{y} \\ &\Rightarrow \boldsymbol{\beta} = (\mathbf{X}'\mathbf{Z}\mathbf{Z}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Z}\mathbf{Z}'\mathbf{y} \text{ [does not simplify further as } m > k] \end{aligned}$$

4.(a) We have $\widehat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{A}\mathbf{X})^{-1}\mathbf{X}'\mathbf{A}(\mathbf{X}\boldsymbol{\beta} + \mathbf{u}) = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{A}\mathbf{X})^{-1}\mathbf{X}'\mathbf{A}\mathbf{u}$.
So $\mathrm{E}[\widehat{\boldsymbol{\beta}}] = \boldsymbol{\beta} + \mathrm{E}[(\mathbf{X}'\mathbf{A}\mathbf{X})^{-1}\mathbf{X}'\mathbf{A}\mathbf{u}] = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{A}\mathbf{X})^{-1}\mathbf{X}'\mathbf{A}\mathrm{E}[\mathbf{u}] = \boldsymbol{\beta}$, as $\mathrm{E}[\mathbf{u}] = \mathbf{0}$.
And $\mathrm{V}[\widehat{\boldsymbol{\beta}}] = \mathrm{E}[(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})'] = \mathrm{E}[((\mathbf{X}'\mathbf{A}\mathbf{X})^{-1}\mathbf{X}'\mathbf{A}\mathbf{u}) \times ((\mathbf{X}'\mathbf{A}\mathbf{X})^{-1}\mathbf{X}'\mathbf{A}\mathbf{u})']$
$= (\mathbf{X}'\mathbf{A}\mathbf{X})^{-1}\mathbf{X}'\mathbf{A}\mathrm{E}[\mathbf{u}\mathbf{u}']\mathbf{X}\mathbf{A}(\mathbf{X}'\mathbf{A}\mathbf{X})^{-1} = (\mathbf{X}'\mathbf{A}\mathbf{X})^{-1}\mathbf{X}'\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\mathbf{X}(\mathbf{X}'\mathbf{A}\mathbf{X})^{-1}$.

(b) We have

$$\begin{aligned} \widehat{\boldsymbol{\beta}} &= \boldsymbol{\beta} + \left(N^{-1}\mathbf{X}'\mathbf{A}\mathbf{X}\right)^{-1} N^{-1}\mathbf{X}'\mathbf{A}\mathbf{u} \\ &\xrightarrow{p} \boldsymbol{\beta} + \left(\operatorname{plim} N^{-1}\mathbf{X}'\mathbf{A}\mathbf{X}\right)^{-1} \operatorname{plim} N^{-1}\mathbf{X}'\mathbf{A}\mathbf{u} \\ &\xrightarrow{p} \boldsymbol{\beta} \text{ since first plim is finite and second is zero.} \end{aligned}$$

(c) We have

$$\begin{aligned} \sqrt{N}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) &= \left(N^{-1}\mathbf{X}'\mathbf{A}\mathbf{X}\right)^{-1} \frac{1}{\sqrt{N}}\mathbf{X}'\mathbf{A}\mathbf{u} \\ &\xrightarrow{d} \left(\operatorname{plim} N^{-1}\mathbf{X}'\mathbf{A}\mathbf{X}\right)^{-1} \times \mathcal{N}[\mathbf{0},\ \mathbf{B}] \\ &\xrightarrow{p} \mathcal{N}\left[\mathbf{0}, \left(\operatorname{plim} N^{-1}\mathbf{X}'\mathbf{A}\mathbf{X}\right)^{-1} \ \mathbf{B} \left(\operatorname{plim} N^{-1}\mathbf{X}'\mathbf{A}\mathbf{X}\right)^{-1}\right] \end{aligned}$$

(d) Here

$$\begin{aligned} \mathbf{B} &= \lim \mathrm{V}\left[\frac{1}{\sqrt{N}}\mathbf{X}'\mathbf{A}\mathbf{u}\right] = \lim \mathrm{E}\left[\frac{1}{N}\mathbf{X}'\mathbf{A}\mathbf{u}\mathbf{u}'\mathbf{A}\mathbf{X}\right] = \lim N^{-1}\mathbf{X}'\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\mathbf{X}. \\ \Rightarrow\ & \widehat{\boldsymbol{\beta}} \overset{a}{\sim} \mathcal{N}[\boldsymbol{\beta}, (\mathbf{X}'\mathbf{A}\mathbf{X})^{-1}\mathbf{X}'\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\mathbf{X}(\mathbf{X}'\mathbf{A}\mathbf{X})^{-1}]. \end{aligned}$$

5.(a) Here $\widehat{y} = 350.8251 + 0.0017691 \times 179420.7 = 668.2382 = \overline{value}$.
Not surprised. Since OLS residuals sum to zero, $\overline{\widehat{y}} = \overline{y + \widehat{u}} = \overline{y} + \overline{\widehat{u}} = \overline{y}$.

(b) The claim is the alternative, so we reject if $\beta_{\text{hhsize}} > 0$ and here $\widehat{\beta}_{\text{hhsize}} = 69.27 > 0$.
The p-value for a one-sided test is half that for two-sided test: $0.054/2 = 0.027$.
Since $p = 0.027 < 0.05$ we reject $H_0: \ \beta_{\text{hhsize}} \leq 0$ at level 0.05 and confirm the claim.

(c) This is not clear. The R^2 increases from 0.8983 to 0.9194, though we should adjust for degrees of freedom and this is not given here [Stata does not report $\overline{R}^2$ when the robust option is used, though we could calculate it given the reported root MSE and standard deviation of rent.] Vacrate is clearly statistically insignificant at 5%, hhsize is borderline, and percrent is clearly statistically significant at 5%.

(d)(i) The Stata command is `test vacrate hhsize percrent`
[Note that if errors are heteroskedastic then we cannot use the usual F-test in terms of sums of squared residuals. For this reason Stata did not give the ANOVA table when the robust option was used in regress. Instead we use $\widehat{\boldsymbol{\beta}} \overset{a}{\sim} \mathcal{N}[\mathbf{0}, \mathbf{V}]$ gives $W = (\mathbf{R}\widehat{\boldsymbol{\beta}} - \mathbf{r})'\widehat{\mathbf{V}}^{-1}(\mathbf{R}\widehat{\boldsymbol{\beta}} - \mathbf{r}) \overset{a}{\sim} \chi^2(q)$ under $H_0: \mathbf{R}\boldsymbol{\beta} - \mathbf{r} = \mathbf{0}$, where $\widehat{\mathbf{V}}$ is the heteroskedastic robust estimate. The Stata command `test` gives an F-version of this $F = W/q$].
(ii) The Stata command is `regress rent value, robust`

(e) The third equation directly gives the elasticity.
$t = (\widehat{\beta} - 1)/s_{\widehat{\beta}} = (0.5149396 - 1)/0.0208438 = -23.27$. $|t| > t_{56;.025} \simeq 2$.
Very strong rejection of $H_0: \beta = 1$ against $H_a: \beta \neq 1$.

(f)(i) Plot rent against value along with an OLS regression line and see of variability around the line increases as value increases.
(ii) See whether there is a big difference between heteroskedastic-robust standard errors and standard errors that assume homoskedastic errors.

(g) Run the OLS regression

$$\frac{\text{rent}}{\text{value}} = \beta_1 \frac{1}{\text{value}} + \beta_2 \frac{\text{value}}{\text{value}} + u^*$$

since the error $u^* = u/\text{value} \sim [0, \sigma^2]$ if $u \sim [0, \sigma^2 \text{value}^2]$.

	Exam / 50		Exam / 50		
75*th* percentile	43 (86%)			B+	28 and above
Median	38.75 (77.5%)	A	43 and above	B	20.5 and above
25*th* percentile	34.5 (59%)	A-	35.5 and above		