240D Fall 2005 Solutions to Midterm Exam

1.(a) Here

$$\ln f(y) = \ln 2 - 2\ln\gamma + \ln y - \left(\frac{y}{\gamma}\right)^2 \text{ and } \gamma = \exp(\mathbf{x}'\boldsymbol{\beta}) \text{ and } \ln\gamma = \mathbf{x}'\boldsymbol{\beta}$$

$$\Rightarrow Q_N(\boldsymbol{\beta}) = \frac{1}{N}\sum_{i=1}^N \ln f(y_i) = \frac{1}{N}\sum_{i=1}^N \left\{\ln 2 - 2\mathbf{x}_i'\boldsymbol{\beta} + \ln y_i - \left(\frac{y_i}{\exp(\mathbf{x}_i'\boldsymbol{\beta})}\right)^2\right\}.$$

(b) Now

$$\begin{aligned} Q_0(\boldsymbol{\beta}) &= \operatorname{plim} Q_N(\boldsymbol{\beta}) \\ &= \ln 2 - 2\lim\frac{1}{N}\sum_{i=1}^N \mathbf{x}_i'\boldsymbol{\beta} + \lim\frac{1}{N}\sum_{i=1}^N \mathrm{E}[\ln y_i] - \lim\frac{1}{N}\sum_{i=1}^N \frac{\mathrm{E}[y_i^2]}{(\exp(\mathbf{x}_i'\boldsymbol{\beta}))^2} \\ &= \ln 2 - 2\lim\frac{1}{N}\sum_{i=1}^N \mathbf{x}_i'\boldsymbol{\beta} + \lim\frac{1}{N}\sum_{i=1}^N \left(\mathbf{x}_i'\boldsymbol{\beta}_0 - \frac{c}{2}\right) - \lim\frac{1}{N}\sum_{i=1}^N \frac{(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2}{(\exp(\mathbf{x}_i'\boldsymbol{\beta}))^2} \end{aligned}$$

using $\mathrm{E}[\ln y_i] = \ln\gamma_{i0} - c/2 = \ln(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0)) - c/2 = \mathbf{x}_i'\boldsymbol{\beta}_0 - c/2$ and $\mathrm{E}[y_i^2] = \gamma_{i0}^2 = (\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2$.

(c) Differentiate wrt $\boldsymbol{\beta}$ (not $\boldsymbol{\beta}_0$)

$$\begin{aligned} \frac{\partial Q_0(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}} &= -2\lim\frac{1}{N}\sum_{i=1}^N \mathbf{x}_i + \lim\frac{1}{N}\sum_{i=1}^N 2\left(\frac{\exp(\mathbf{x}_i'\boldsymbol{\beta}_0)}{\exp(\mathbf{x}_i'\boldsymbol{\beta})}\right)^2 \mathbf{x}_i \\ &= \mathbf{0} \text{ when } \boldsymbol{\beta} = \boldsymbol{\beta}_0. \end{aligned}$$

[Also $\partial^2 Q_0(\boldsymbol{\beta})/\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}' = -\lim N^{-1}\sum_i 4\left(\frac{\exp(\mathbf{x}_i'\boldsymbol{\beta}_0)}{\exp(\mathbf{x}_i'\boldsymbol{\beta})}\right)^2 \mathbf{x}_i\mathbf{x}_i'$ is negative definite at $\boldsymbol{\beta}_0$.]
Since $\operatorname{plim} Q_N(\boldsymbol{\beta})$ attains a local maximum at $\boldsymbol{\beta} = \boldsymbol{\beta}_0$, conclude that $\widehat{\boldsymbol{\beta}} = \arg\max Q_N(\boldsymbol{\beta})$ is consistent for $\boldsymbol{\beta}_0$.

(d) Since x_i is nonstochastic and varies over i, the quantities being averaged such as $\ln y_i$ and $y_i^2/\exp(\mathbf{x}_i'\boldsymbol{\beta})^2$ are inid so need to use Markov LLN.
This requires existence of higher order moments such as second moments of $\ln y_i$ and y_i^2.
[Note that need tosay what we are applying Markov LLN to].

2.(a) Differentiating

$$\begin{aligned} \frac{\partial Q_N}{\partial\boldsymbol{\beta}} &= \frac{1}{N}\sum_{i=1}^N \left(-2\mathbf{x}_i + 2\frac{y_i^2}{(\exp(\mathbf{x}_i'\boldsymbol{\beta}))^2}\mathbf{x}_i\right) \\ &= \frac{1}{N}\sum_{i=1}^N 2\times\frac{y_i^2 - (\exp(\mathbf{x}_i'\boldsymbol{\beta}))^2}{(\exp(\mathbf{x}_i'\boldsymbol{\beta}))^2}\mathbf{x}_i \text{ upon rearranging.} \end{aligned}$$

So $g(\mathbf{x}_i'\boldsymbol{\beta}) = (\exp(\mathbf{x}_i'\boldsymbol{\beta}))^2$.

(b) Since

$$\lim \mathrm{E}\left[\left.\frac{\partial Q_N}{\partial\boldsymbol{\beta}}\right|_{\boldsymbol{\beta}_0}\right] = 0 \text{ if } \mathrm{E}[y_i^2|\mathbf{x}_i] = (\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2,$$

the essential condition is correct specification of $\mathrm{E}[y_i^2|\mathbf{x}_i]$.

(c) Can assume the result that $\sqrt{N}\left.\frac{\partial Q_N}{\partial \boldsymbol{\beta}}\right|_{\boldsymbol{\beta}_0} \xrightarrow{d} \mathcal{N}[\mathbf{0}, \mathbf{B}_0]$, where

$$\begin{aligned}
\mathbf{B}_0 &= \operatorname{plim} N \left.\frac{\partial Q_N}{\partial \boldsymbol{\beta}}\frac{\partial Q_N}{\partial \boldsymbol{\beta}'}\right|_{\boldsymbol{\beta}_0} \\
&= \operatorname{plim} N\left(\frac{1}{N}\sum_{i=1}^N 2\times\frac{y_i^2-(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2}{(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2}\mathbf{x}_i\right)\left(\frac{1}{N}\sum_{i=1}^N 2\times\frac{y_i^2-(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2}{(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2}\mathbf{x}_i\right)' \\
&= \lim \frac{1}{N}\sum_{i=1}^N \mathrm{E}\left[\left(2\times\frac{y_i^2-(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2}{(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2}\right)^2\mathbf{x}_i\mathbf{x}_i'\right] \text{ using independence and } \mathrm{E}[y_i^2-(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2]=0 \\
&= \lim \frac{1}{N}\sum_{i=1}^N 4\times\frac{\mathrm{E}[\{y_i^2-(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2\}^2]}{(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^4}\mathbf{x}_i\mathbf{x}_i' \\
&= \lim \frac{1}{N}\sum_{i=1}^N 4\times\frac{\mathrm{V}[y_i^2]}{(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^4}\mathbf{x}_i\mathbf{x}_i' \\
&= \lim \frac{1}{N}\sum_{i=1}^N 4\mathbf{x}_i\mathbf{x}_i' \qquad \text{as } \mathrm{V}[y_i^2]=(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^4
\end{aligned}$$

Alternatively can apply CLT to average of the term in the sum.
Now y_i^2 has mean $(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2$ and variance $(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^4$.
So $2\times\frac{y_i^2-(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2}{(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2}\mathbf{x}_i$ has mean $\mathbf{0}$ and variance $4\times\frac{(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^4}{(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^4}\mathbf{x}_i\mathbf{x}_i' = 4\mathbf{x}_i\mathbf{x}_i'$.
Thus

$$\begin{aligned}
Z_N &= \left(\frac{1}{N}\sum_{i=1}^N 4\mathbf{x}_i\mathbf{x}_i'\right)^{-1/2}\times\left(\frac{1}{\sqrt{N}}\sum_{i=1}^N 2\times\frac{y_i^2-(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2}{(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2}\mathbf{x}_i\right)\xrightarrow{d}\mathcal{N}[\mathbf{0},\mathbf{I}] \\
&\Rightarrow \frac{1}{\sqrt{N}}\sum_{i=1}^N 2\times\frac{y_i^2-(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2}{(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2}\mathbf{x}_i\xrightarrow{d}\mathcal{N}\left[\mathbf{0},\ \lim\frac{1}{N}\sum_{i=1}^N 4\mathbf{x}_i\mathbf{x}_i'\right]
\end{aligned}$$

(d)

$$\left.\frac{\partial^2 Q_N}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'}\right|_{\boldsymbol{\beta}_0} = \frac{1}{N}\sum_{i=1}^N\left(-4\frac{y_i^2}{(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2}\mathbf{x}_i\mathbf{x}_i'\right)\xrightarrow{p}\lim\frac{1}{N}\sum_{i=1}^N -4\mathbf{x}_i\mathbf{x}_i', \qquad \text{since } \mathrm{E}[y_i^2]=(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2.$$

(e)

$$\begin{aligned}
&\sqrt{N}(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0)\xrightarrow{d}\mathcal{N}[\mathbf{0},\mathbf{A}(\boldsymbol{\beta}_0)^{-1}\mathbf{B}(\boldsymbol{\beta}_0)\mathbf{A}(\boldsymbol{\beta}_0)^{-1}] \\
&\xrightarrow{d}\mathcal{N}\left[\mathbf{0},\left(\lim\frac{1}{N}\sum_{i=1}^N -4\mathbf{x}_i\mathbf{x}_i'\right)^{-1}\left(\lim\frac{1}{N}\sum_{i=1}^N 4\mathbf{x}_i\mathbf{x}_i'\right)\left(\lim\frac{1}{N}\sum_{i=1}^N -4\mathbf{x}_i\mathbf{x}_i'\right)^{-1}\right] \\
&\xrightarrow{d}\mathcal{N}\left[\mathbf{0},\left(\lim\frac{1}{N}\sum_{i=1}^N 4\mathbf{x}_i\mathbf{x}_i'\right)^{-1}\right].
\end{aligned}$$

3. For this problem

$$\mathbf{H} = \frac{\partial^2 Q_N(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'} = \frac{1}{N}\sum_{i=1}^{N}\left(-4\frac{y_i^2}{(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2}\mathbf{x}_i\mathbf{x}_i'\right)$$

$$\Rightarrow \ \mathbf{EH}_s = \mathrm{E}\left[\mathbf{H}|_{\boldsymbol{\beta}_0}\right]\Big|_{\widehat{\boldsymbol{\beta}}_s} = \frac{1}{N}\sum_{i=1}^{N}-4\mathbf{x}_i\mathbf{x}_i', \quad \text{since } \mathrm{E}[y_i^2] = (\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2$$

and

$$\mathbf{g}_s = \left.\frac{\partial Q_N(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}}\right|_{\widehat{\boldsymbol{\beta}}_s} = \frac{1}{N}\sum_{i=1}^{N}2\times\frac{y_i^2-(\exp(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}_s))^2}{(\exp(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}_s))^2}\mathbf{x}_i$$

Using Newton-Raphson the update rule is

$$(\widehat{\boldsymbol{\beta}}_{s+1}-\widehat{\boldsymbol{\beta}}_s) = -\mathbf{H}_s^{-1}\mathbf{g}_s = \left(\frac{1}{N}\sum_{i=1}^{N}-4\frac{y_i^2}{(\exp(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}))^2}\mathbf{x}_i\mathbf{x}_i'\right)^{-1}\frac{1}{N}\sum_{i=1}^{N}2\times\frac{y_i^2-(\exp(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}_s))^2}{(\exp(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}_s))^2}\mathbf{x}_i.$$

Or using method-of-scoring the update rule is

$$(\widehat{\boldsymbol{\beta}}_{s+1}-\widehat{\boldsymbol{\beta}}_s) = -\mathbf{EH}_s^{-1}\mathbf{g}_s = \left(\frac{1}{N}\sum_{i=1}^{N}4\mathbf{x}_i\mathbf{x}_i'\right)^{-1}\frac{1}{N}\sum_{i=1}^{N}2\times\frac{y_i^2-(\exp(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}_s))^2}{(\exp(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}_s))^2}\mathbf{x}_i$$

[Very easy as thus is OLS coefficient from regress $\frac{y_i^2-(\exp(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}_s))^2}{(\exp(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}_s))^2}$ on $2\mathbf{x}_i$.]

For a starting value for $\boldsymbol{\beta}$ can use either:
(1) $\mathrm{E}[y_i] = (\exp(\mathbf{x}_i'\boldsymbol{\beta}))\Gamma(1.5) \Rightarrow \ln\mathrm{E}[y_i] - \ln\Gamma(1.5) = \mathbf{x}_i'\boldsymbol{\beta}$, so do OLS of $\ln y_i - \ln\Gamma(1.5)$ on $\mathbf{x}_i$; or
(2) $\mathrm{E}[y_i^2] = (\exp(\mathbf{x}_i'\boldsymbol{\beta}))^2 \Rightarrow \ln\mathrm{E}[y_i^2] = 2\mathbf{x}_i'\boldsymbol{\beta}$, so do OLS of $\ln y_i^2$ on $\mathbf{x}_i$. This suggests OLS regression of $\ln y_i^2$ on $\mathbf{x}_i$.
[Note that this will not give consistent estimate, however, since e.g. $\ln\mathrm{E}[y_i] \neq \mathrm{E}[\ln y_i]$].
[Note that OLS of y_i on $\mathbf{x}_i$ not appropriate as $\mathrm{E}[y_i|\mathbf{x}_i] \neq \mathbf{x}_i'\boldsymbol{\beta}$].

4.(a) $H_0 : \beta_2 = 1$ against $H_a : \beta_2 = 1$.
Use $t = (\widehat{\beta}_2 - 1)/\mathrm{se}[\widehat{\beta}_2] = (1.6-1)/0.4 = 1.5$.
Since $|t| \not> z_{.025} = 1.96$ we do not reject H_0 at level 0.05.

(b) For the logit model

$$\frac{\partial\Pr[y|\mathbf{x}]}{\partial\mathbf{x}} = \frac{\partial\Lambda(\mathbf{x}'\boldsymbol{\beta})}{\partial\mathbf{x}} = \Lambda(\mathbf{x}'\boldsymbol{\beta})(1-\Lambda(\mathbf{x}'\boldsymbol{\beta}))\boldsymbol{\beta}.$$

One possibility is to evaluate at $\mathbf{x} = \bar{\mathbf{x}}$. Here $\bar{\mathbf{x}}'\widehat{\boldsymbol{\beta}} = 0.2+1.6\times\bar{x} = 0.2+1.6\times0.5 = 1.0$. So $\Lambda(\bar{\mathbf{x}}'\widehat{\boldsymbol{\beta}}) = e^1/(1+e^1) = 2.7183/3.7183 = 0.731$. The marginal effect is then $0.731\times(1-0.731)\times1.6 = 0.315$. Another possibility is to evaluate at the sample average probability, letting $\Lambda(\mathbf{x}'\boldsymbol{\beta}) = \bar{y} = 0.75$. The marginal effect is then $0.75\times(1-0.75)\times1.6 = 0.3$.

(c) The logit first-order conditions are

$$\frac{1}{N}\sum_{i=1}^{N}(y_i - \Lambda(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}))\mathbf{x}_i = 0.$$

Since $\mathbf{x}_i$ includes the intercept this implies that $\frac{1}{N}\sum_{i=1}^{N}(y_i - \Lambda(\mathbf{x}_i'\widehat{\boldsymbol{\beta}})) = 0$, so that $\frac{1}{N}\sum_{i=1}^{N}\Lambda(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}) = \bar{y}$. For a logit model with intercept the average of fitted probabilities always equals the sample frequency $\bar{y}$.
Instead consider other criteria such as spread of predicted probabilities and pseudo-R^2.
[Almost no-one got this].

(d) The probit model has a quite different functional form, $p_i = \Phi(\mathbf{x}_i'\boldsymbol{\beta})$ rather than $\Lambda(\mathbf{x}_i'\boldsymbol{\beta})$, so the parameters $\boldsymbol{\beta}$ are not comparable across models and will differ greatly in magnitude. The predicted probabilities and marginal effects, however, may be similar across models.
[Many missed this].

(f) Estimate the logit model by NLS regression of the binary dependent variable y_i on $\Lambda(\mathbf{x}_i'\boldsymbol{\beta})$, i.e. run the regression

$$y_i = \Lambda(\mathbf{x}_i'\boldsymbol{\beta}) + u_i.$$

For valid standard errors obtain heteroskedastic-robust standard errors.

The course grade will be based on a curve from the combined scores of midterm 1 (35%), final (50%) and assignments (15%).
The curve for this exam is only a guide to give you a rough idea of how you are doing.

Scores out of	35		A	28 and above
$75th$ percentile	29	(83%)	A-	24 and above
Median	27	(77%)	B+	20 and above
$25th$ percentile	24	(69%)	B	16 and above