# 240D Fall 2008 Solutions to Midterm Exam

**1.(a)** Here

$$\begin{aligned} \ln f(y) &= \ln \lambda - y\lambda \text{ and } \lambda = \exp(\mathbf{x}'\boldsymbol{\beta}) \text{ and } \ln \lambda = \mathbf{x}'\boldsymbol{\beta} \\ &\Rightarrow Q_N(\boldsymbol{\beta}) = \frac{1}{N}\sum\nolimits_i \ln f(y_i) = \frac{1}{N}\sum\nolimits_i \{\mathbf{x}_i'\boldsymbol{\beta} - y_i \exp(\mathbf{x}_i'\boldsymbol{\beta})\}. \end{aligned}$$

**(b)** Now

$$\begin{aligned} Q_0(\boldsymbol{\beta}) &= \operatorname{plim} Q_N(\boldsymbol{\beta}) \\ &= \operatorname{plim} \frac{1}{N}\sum\nolimits_i \mathbf{x}_i'\boldsymbol{\beta} - \operatorname{plim} \frac{1}{N}\sum\nolimits_i y_i \exp(\mathbf{x}_i'\boldsymbol{\beta}) \\ &= \lim \frac{1}{N}\sum\nolimits_i \mathbf{x}_i'\boldsymbol{\beta} - \lim \frac{1}{N}\sum\nolimits_i \mathrm{E}[y_i \exp(\mathbf{x}_i'\boldsymbol{\beta})] \quad \text{if LLN can be applied} \\ &= \lim \frac{1}{N}\sum\nolimits_i \mathbf{x}_i'\boldsymbol{\beta} - \lim \frac{1}{N}\sum\nolimits_i \exp(-\mathbf{x}_i'\boldsymbol{\beta}_0) \times \exp(\mathbf{x}_i'\boldsymbol{\beta}) \text{ if } \mathrm{E}[y_i|\mathbf{x}_i] = \exp(-\mathbf{x}_i'\boldsymbol{\beta}_0) \end{aligned}$$

**(c)** Differentiate w.r.t. $\boldsymbol{\beta}$ (not $\boldsymbol{\beta}_0$)

$$\begin{aligned} \frac{\partial Q_0(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} &= \lim \frac{1}{N}\sum\nolimits_i \mathbf{x}_i - \lim \frac{1}{N}\sum\nolimits_i \exp(-\mathbf{x}_i'\boldsymbol{\beta}_0)\exp(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i \\ &= \mathbf{0} \text{ when } \boldsymbol{\beta} = \boldsymbol{\beta}_0. \end{aligned}$$

Since $\operatorname{plim} Q_N(\boldsymbol{\beta})$ attains a local maximum at $\boldsymbol{\beta} = \boldsymbol{\beta}_0$,
conclude that $\widehat{\boldsymbol{\beta}} = \arg\max Q_N(\boldsymbol{\beta})$ is consistent for $\boldsymbol{\beta}_0$.

[Also formally but not required $\partial^2 Q_0(\boldsymbol{\beta})/\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}' = \lim N^{-1}\sum_i (-\exp(-\mathbf{x}_i'\boldsymbol{\beta}_0) \times \exp(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i\mathbf{x}_i')$
$= \lim N^{-1}\sum_i -\mathbf{x}_i\mathbf{x}_i'$ at $\boldsymbol{\beta}_0$ is negative definite.]

**(d)** Consider the second term: $\operatorname{plim} \frac{1}{N}\sum_i y_i \exp(\mathbf{x}_i'\boldsymbol{\beta})$.
This is the average $\overline{X}_N = N^{-1}\sum_i X_i$ where $X_i = y_i \exp(\mathbf{x}_i'\boldsymbol{\beta})$.
Here $X_i = y_i \exp(\mathbf{x}_i'\boldsymbol{\beta})$ is i.n.i.d. So need Markov LLN.
With $\delta = 1$ this requires that $\mathrm{V}[y_i|\mathbf{x}_i]$ exists and $\mathbf{x}_i$ is bounded.
[More formally, with $\delta = 1$ we need $\sum_{i=1}^{\infty}\mathrm{E}[\{(y_i \exp(\mathbf{x}_i'\boldsymbol{\beta}) - \mathrm{E}[y_i \exp(\mathbf{x}_i'\boldsymbol{\beta})]\}^2]/i^{\delta} = \sum_{i=1}^{\infty}\mathrm{V}[y_i]\exp(2\mathbf{x}_i'\boldsymbol{\beta})/i^{\delta} < \infty$.]

**(e)** From (c) and (d) all we need for consistency is $\mathrm{E}[y_i|\mathbf{x}_i] = \exp(\mathbf{x}_i'\boldsymbol{\beta}_0)$.
Provided this holds the MLE is consistent even if other aspects of the distribution are misspecified.
If $\mathrm{E}[y_i|\mathbf{x}_i] \neq \exp(\mathbf{x}_i'\boldsymbol{\beta}_0)$ then the MLE is inconsistent.

**2.(a)** Differentiating

$$\begin{aligned}\frac{\partial Q_N}{\partial \boldsymbol{\beta}} &= \frac{1}{N}\sum\nolimits_i (\mathbf{x}_i - y_i \exp(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i) \\ &= \frac{1}{N}\sum\nolimits_i (\exp(-\mathbf{x}_i'\boldsymbol{\beta})\exp(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i - y_i \exp(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i) \\ &= -\frac{1}{N}\sum\nolimits_i (y_i - \exp(-\mathbf{x}_i'\boldsymbol{\beta}))\exp(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i \quad \text{simplifying}\end{aligned}$$

**(b)** Can assume the result that $\sqrt{N}\left.\frac{\partial Q_N}{\partial \boldsymbol{\beta}}\right|_{\boldsymbol{\beta}_0} \xrightarrow{d} \mathcal{N}[\mathbf{0},\, \mathbf{B}_0]$, where

$$\begin{aligned}\mathbf{B}_0 &= \operatorname{plim} N \left.\frac{\partial Q_N}{\partial \boldsymbol{\beta}}\frac{\partial Q_N}{\partial \boldsymbol{\beta}'}\right|_{\boldsymbol{\beta}_0} \\ &= \operatorname{plim} N \left(\frac{1}{N}\sum\nolimits_i (y_i - \exp(-\mathbf{x}_i'\boldsymbol{\beta}_0))\exp(\mathbf{x}_i'\boldsymbol{\beta}_0)\mathbf{x}_i\right)\left(\frac{1}{N}\sum\nolimits_i (y_i - \exp(-\mathbf{x}_i'\boldsymbol{\beta}_0))\exp(\mathbf{x}_i'\boldsymbol{\beta}_0)\mathbf{x}_i\right)' \\ &= \lim \frac{1}{N}\sum\nolimits_i \mathrm{E}\left[(y_i - \exp(-\mathbf{x}_i'\boldsymbol{\beta}_0))^2(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2\mathbf{x}_i\mathbf{x}_i'\right] \text{ using independence and } \mathrm{E}[y_i - \exp(-\mathbf{x}_i'\boldsymbol{\beta}_0)] = 0 \\ &= \lim \frac{1}{N}\sum\nolimits_i \mathrm{V}[y_i|\mathbf{x}_i](\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2\mathbf{x}_i\mathbf{x}_i' \\ &= \lim \frac{1}{N}\sum\nolimits_i \frac{\exp(\mathbf{x}_i'\boldsymbol{\beta}_0)^2}{\exp(\mathbf{x}_i'\boldsymbol{\beta}_0)^2}\mathbf{x}_i\mathbf{x}_i' \qquad \text{as } \mathrm{V}[y_i] = 1/(\exp(\mathbf{x}_i'\boldsymbol{\beta}_0))^2 \\ &= \lim \frac{1}{N}\sum\nolimits_{i=1}^N \mathbf{x}_i\mathbf{x}_i'\end{aligned}$$

**(c)** We have

$$\begin{aligned}\frac{\partial^2 Q_N}{\partial \boldsymbol{\beta}\partial \boldsymbol{\beta}'} &= \frac{1}{N}\sum\nolimits_i -y_i\exp(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i\mathbf{x}_i' \text{ where derivative w.r.t. first expression in part (a)} \\ \mathbf{A}_0 &= \operatorname{plim}\left.\frac{\partial^2 Q_N}{\partial \boldsymbol{\beta}\partial \boldsymbol{\beta}'}\right|_{\boldsymbol{\beta}_0} = \lim \frac{1}{N}\sum\nolimits_i -\exp(\mathbf{x}_i'\boldsymbol{\beta}_0)\exp(\mathbf{x}_i'\boldsymbol{\beta}_0)\mathbf{x}_i\mathbf{x}_i', \quad \text{since } \mathrm{E}[y_i] = \exp(-\mathbf{x}_i'\boldsymbol{\beta}_0). \\ \mathbf{A}_0 &= -\lim \frac{1}{N}\sum_{i=1}^N \mathbf{x}_i\mathbf{x}_i' \text{ simplifying}\end{aligned}$$

**(d)** Combining

$$\begin{aligned}&\sqrt{N}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) \xrightarrow{d} \mathcal{N}[\mathbf{0},\ \mathbf{A}_0{}^{-1}\mathbf{B}_0\mathbf{A}_0{}^{-1}] \\ &\xrightarrow{d} \mathcal{N}[\mathbf{0},\ -\mathbf{A}_0^{-1}] \text{ since from above } \mathbf{A}_0 = -\mathbf{B}_0 \\ &\xrightarrow{d} \mathcal{N}\left[\mathbf{0},\ \left(\lim \frac{1}{N}\sum\nolimits_{i=1}^N \mathbf{x}_i\mathbf{x}_i'\right)^{-1}\right].\end{aligned}$$

**(e)** Reject $H_0 : \beta_2 = 0$ against $H_a : \beta_2 \neq 0$ at level $a$ if $|t| > 1.96$ where $t = \widehat{\beta}_j/s_{\widehat{\beta}_j}$ where $s^2_{\widehat{\beta}_j}$ is the $j^{th}$ diagonal entry in $\left(\sum_{i=1}^N \mathbf{x}_i\mathbf{x}_i'\right)^{-1}$.

**3.(a)** Simplest is Lindeberg-Levy CLT.
Let $\{X_i\}$ be iid with $\mathrm{E}[X_i] = \mu$ and $\mathrm{V}[X_i] = \sigma^2$. Then $Z_N = \frac{\bar{X}_N - \mu}{\sigma/\sqrt{N}} \xrightarrow{d} \mathcal{N}[0, 1]$.
[Other CLT's can be given].

**(b)** The limit distribution is $\sqrt{N}(\widehat{\boldsymbol{\beta}}_{\text{NLS}} - \boldsymbol{\beta}_0) \xrightarrow{d} \mathcal{N}\left[\mathbf{0}, \mathbf{A}_0^{-1}\mathbf{B}_0\mathbf{A}_0^{-1}\right]$
where $\mathbf{A}_0 = \text{plim}\frac{1}{N}\sum_{i=1}^N \sum_{j=1}^N \frac{\partial g_i}{\partial \boldsymbol{\beta}}\frac{\partial g_j}{\partial \boldsymbol{\beta}'}\Big|_{\boldsymbol{\beta}_0}$ and $\mathbf{B}_0 = \text{plim}\frac{1}{N}\sum_{i=1}^N u_i^2 \frac{\partial g_i}{\partial \boldsymbol{\beta}}\frac{\partial g_j}{\partial \boldsymbol{\beta}'}\Big|_{\boldsymbol{\beta}_0}$ and $g_i = \mathrm{E}[y_i|\mathbf{x}_i]$.
Asymptotically

$$\widehat{\boldsymbol{\beta}}_{\text{NLS}} \overset{a}{\sim} \mathcal{N}\left[\boldsymbol{\beta},\, \widehat{\mathbf{A}}^{-1}\widehat{\mathbf{B}}\widehat{\mathbf{A}}^{-1}\right]$$

$$\widehat{\mathbf{A}} = \frac{1}{N}\sum_{i=1}^N \frac{\partial g_i}{\partial \boldsymbol{\beta}}\Big|_{\widehat{\boldsymbol{\beta}}} \frac{\partial g_i}{\partial \boldsymbol{\beta}'}\Big|_{\widehat{\boldsymbol{\beta}}}$$

$$\widehat{\mathbf{B}} = \frac{1}{N}\sum_{i=1}^N \widehat{u}_i^2 \frac{\partial g_i}{\partial \boldsymbol{\beta}}\Big|_{\widehat{\boldsymbol{\beta}}} \frac{\partial g_i}{\partial \boldsymbol{\beta}'}\Big|_{\widehat{\boldsymbol{\beta}}}$$

**(c)** Using Newton-Raphson the update rule is

$$(\widehat{\boldsymbol{\theta}}_{s+1} - \widehat{\boldsymbol{\theta}}_s) = -\left[\frac{\partial \mathbf{g}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'}\Big|_{\widehat{\boldsymbol{\theta}}_s}\right]^{-1} \mathbf{g}(\widehat{\boldsymbol{\theta}}_s).$$

**4.(a)** Since Poisson sets $\mathrm{E}[y|\mathbf{x}] = \exp(\mathbf{x}'\boldsymbol{\beta})$, $\beta_j$ is a semi-elasticity.
A \$1,000 increase in income (which is a one unit change) is associated with a $100 \times 0.003557 = 0.3557$ percent increase in the expected number of doctor visits.

**(b)** Since Poisson sets $\mathrm{E}[y|\mathbf{x}] = \exp(\mathbf{x}_i'\boldsymbol{\beta})$, the marginal effect is $\partial\mathrm{E}[y|\mathbf{x}]/\partial x_j = \exp(\mathbf{x}_i'\boldsymbol{\beta}) \times \beta_j$.
The estimated marginal effect is therefore
$\exp(\mathbf{x}^{*\prime}\widehat{\boldsymbol{\beta}}) \times .003557$
$= \exp(1 \times 0.80 + 0 \times 1.09 + 1 \times 0.49 \times 1 + 10 \times .0036 - 0.23) \times .003557$
[The above is enough for full credit]
$= \exp(1.097) \times .003557 = 2.995 \times .003557 = 0.01065$.
The expected number of doctor visits increases by 0.01065.

The course grade will be based on a curve from the combined scores of midterm (35%), final (50%) and assignments (15%).
The curve for this exam is only a guide to give you a rough idea of how you are doing.

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|---|---|---|---|---|
| Scores out of | 35 | | A | 30 and above |
| $75th$ percentile | 31 | (89%) | A- | 26 and above |
| Median | 29 | (83%) | B+ | 22 and above |
| $25th$ percentile | 26 | (74%) | B | 18 and above |