1.(a) We have $x_i'\hat{\beta}$ as prediction of $y_i = x_i'\beta + u_i$. Under classical assumptions including normal error,

\[ y_i - x_i'\hat{\beta} = x_i'\beta + u_i - x_i'\hat{\beta} = x_i'((\hat{\beta} - \beta) + u_i) \sim \mathcal{N}[0, x_i'\sigma^2(X'X)^{-1}x_i + \sigma^2] \]

\[ y_i - x_i'\hat{\beta} \sim \mathcal{N}[0, \sigma^2\{1 + x_i'((X'X)^{-1}x_i)\}] \]

\[ \frac{y_i - x_i'\hat{\beta}}{\sqrt{\sigma^2\{1 + x_i'((X'X)^{-1}x_i)\}}} \sim \mathcal{N}[0, 1] \]

\[ \frac{y_i - x_i'\hat{\beta}}{\sqrt{\sigma^2\{1 + x_i'((X'X)^{-1}x_i)\}}} \sim T_{N-k} \]

\[ 95\% \text{ CI is } \beta \in \hat{\beta} \pm t_{N-k,0.025} \times \sqrt{s^2\{1 + x_i'((X'X)^{-1}x_i)\}}. \]

(b) We have

\[ \hat{\beta}_1 = (X'X)^{-1}X'[X_1'\beta_1 + X_2\beta_2 + u] = \beta_1 + (X_1'X_1)^{-1}X_1X_2\beta_2 + (X_1'X_1)^{-1}X_1u. \]

\[ E[\hat{\beta}_1] = \beta_1 + (X_1'X_1)^{-1}X_1X_2\beta_2 \text{ as } E[u] = 0. \]

Conclude that OLS is biased unless $X_1'X_2 = 0$.

(c) We have

\[ y = \begin{bmatrix} 1 & X_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_2 \end{bmatrix} + u = Z\gamma + u. \]

The usual $(Z'Z)^{-1}Z'y$ becomes

\[ \begin{bmatrix} \hat{\alpha}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} 1' & X_2' \\ 1'X_2' & X_2'X_2' \end{bmatrix}^{-1} \begin{bmatrix} 1'y \\ X_2'y \end{bmatrix} = \begin{bmatrix} N & 0 \\ 0 & X_2'X_2' \end{bmatrix}^{-1} \begin{bmatrix} 1'y \\ X_2'y \end{bmatrix} \]

\[ = \begin{bmatrix} \frac{1}{N} & 0 \\ 0 & (X_2'X_2')^{-1} \end{bmatrix} \begin{bmatrix} 1'y \\ X_2'y \end{bmatrix} = \begin{bmatrix} \frac{1}{N}1'y \\ (X_2'X_2')^{-1}X_2'y \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_1 \\ \hat{\beta}_2 \end{bmatrix} \]

2.(a) We have $\hat{\beta} \sim \mathcal{N}[\beta, \sigma^2(X'X)^{-1}]$, so $R\hat{\beta} - r \sim \mathcal{N}[R\beta - r, \sigma^2R(X'X)^{-1}R']$. Under $H_0$ this simplifies as $R\beta - r = 0$, so $R\hat{\beta} - r \sim \mathcal{N}[0, \sigma^2R(X'X)^{-1}R']$.

Forming the quadratic gives the chi-square test statistic (assuming rank[R] = q)

\[ W = (R\hat{\beta} - r)'[\sigma^2R(X'X)^{-1}R']^{-1}(R\hat{\beta} - r) \sim \chi^2(q) \]

(b) Here $R = [1 \ -2]$ and $r = 0$, so $R\hat{\beta} - r = [1 \ -2][5 \\ 2] = 1$, and

\[ R(X'X)^{-1}R' = R \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -3 & 10 \end{bmatrix} \]

So $W = (1)'[0.1 \times 10]^{-1}(1) = 1$.

The critical value is $\chi^2(0.05) = z_{0.025}^2 = 1.96^2 = 3.84$.

Since $W < 3.84$ we do not reject $H_0: \beta_1 = 2\beta_2$.

Since $q = 1$ here this can also be done as a z-test.

(c) Asymptotically we can replace $\sigma^2$ by a consistent estimate $s^2$ such as $s^2 = \hat{u}'\hat{u}/(N - k)$. Then $W_* = (R\hat{\beta} - r)'[s^2R(X'X)^{-1}R']^{-1}(R\hat{\beta} - r) \sim \chi^2(q)$.

Alternatively can use $F = W_*/q = (R\hat{\beta} - r)'[s^2R(X'X)^{-1}R']^{-1}(R\hat{\beta} - r)/q \sim F(q, N - k)$.
3. Various topics

(a) An adequate answer is that a sequence of random variables $b_N \overset{p}{\rightarrow} b$ if for any $\varepsilon > 0$

$$
\lim_{N \to \infty} \Pr[|b_N - b| < \varepsilon] = 1.
$$

(b) A multivariate central limit places conditions on the vector components $x_i$ of the vector average $\bar{X}_N$ such that

$$(V[\bar{X}_N])^{-1/2} (\bar{X}_N - E[\bar{X}_N]) \overset{d}{\rightarrow} N[0, I].$$

(c) The IV estimator (in the just-identified case) is $\hat{\beta}_V = (Z'X)^{-1}Z'y$ where $Z$ is an $N \times k$ matrix of instruments with the property that $\text{plim} N^{-1}Z'u = 0$.

It has the advantage of being consistent even if OLS is inconsistent due $\text{plim} N^{-1}X'u \neq 0$.

(d) For $u \sim [0, \Sigma]$, $\hat{\beta}_{GLS} = (X'S^{-1}X)^{-1}X'S^{-1}y$.

The key property compared to OLS is that GLS is efficient (BLUE in the linear regression model).

It is also unbiased and consistent whenever GLS is unbiased.

(e) The variance of the OLS estimator is estimated by

$$
\hat{V} = (X'\hat{X})^{-1}X'S(X'X)^{-1}
$$

where $\hat{\Sigma} = \text{Diag}[\hat{u}_i^2] = (\sum_i x_i x_i')^{-1} \sum_i \hat{u}_i^2 x_i (\sum_i x_i x_i')^{-1}$.

(f) We have $(y - X\beta)'ZZ'(y - X\beta) = y'ZZ'y - 2y'ZZ'X\beta + \beta'X'ZZ'X\beta$, so

$$
\Rightarrow \quad \partial(y - X\beta)'ZZ'(y - X\beta)/\partial \beta = -2X'ZZ'y + 2X'ZZ'X\beta = 0
$$

$$
\Rightarrow \quad X'ZZ'X\beta = X'ZZ'y
$$

$$
\Rightarrow \quad \beta = (X'ZZ')^{-1}X'ZZ'y \quad \text{[does not simplify further as } m > k]\]

4.(a) We have $\hat{\beta} = (X'AX)^{-1}X'A(X\beta + u) = \beta + (X'AX)^{-1}X'Au$.

So $E[\hat{\beta}] = \beta + E[(X'AX)^{-1}X'Au] = \beta + (X'AX)^{-1}X'AE[u] = \beta$, as $E[u] = 0$.

And $V[\hat{\beta}] = E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'] = E[((X'AX)^{-1}X'Au)\times((X'AX)^{-1}X'Au)']$

$$
= (X'AX)^{-1}X'AE[uu']XAX(X'AX)^{-1} = (X'AX)^{-1}X'A\Sigma AX(X'AX)^{-1}.
$$

(b) We have

$$
\hat{\beta} = \beta + (N^{-1}X'AX)^{-1}N^{-1}X'Au
$$

$$
\overset{p}{\rightarrow} \beta + (\text{plim} N^{-1}X'AX)^{-1}\text{plim} N^{-1}X'Au
$$

$$
\overset{p}{\rightarrow} \beta \text{ since first plim is finite and second is zero.}
$$

(c) We have

$$
\sqrt{N}(\hat{\beta} - \beta) = (N^{-1}X'AX)^{-1} \frac{1}{\sqrt{N}} X'Au
$$

$$
\overset{d}{\rightarrow} (\text{plim} N^{-1}X'AX)^{-1} \times N[0, B]
$$

$$
\overset{p}{\rightarrow} N[0, (\text{plim} N^{-1}X'AX)^{-1} B (\text{plim} N^{-1}X'AX)^{-1}]
$$

(d) Here

$$
B = \lim V \left[ \frac{1}{\sqrt{N}} X'Au \right] = \lim E \left[ \frac{1}{N} X'Auu'AX \right] = \lim N^{-1}X'A\Sigma AX
$$

$$
\Rightarrow \quad \hat{\beta} \overset{d}{\sim} N[\beta, (X'AX)^{-1}X'A\Sigma AX(X'AX)^{-1}].
$$
(a) Here $\hat{y} = 350.8251 + 0.0017691 \times 179420.7 = 668.2382 = \bar{value}$.
Not surprised. Since OLS residuals sum to zero, $\bar{y} = \bar{\bar{y}} + \bar{\bar{u}} = \bar{\bar{y}}$.

(b) The claim is the alternative, so we reject if $\beta_{hhsiz} > 0$ and here $\hat{\beta}_{hhsiz} = 69.27 > 0$.
The p-value for a one-sided test is half that for two-sided test: $0.054/2 = 0.027$.
Since $p = 0.027 < 0.05$ we reject $H_0 : \beta_{hhsiz} \leq 0$ at level 0.05 and confirm the claim.

(c) This is not clear. The $R^2$ increases from 0.8983 to 0.9194, though we should adjust for degrees of freedom and this is not given here [Stata does not report $R^2$ when the robust option is used, though we could calculate it given the reported root MSE and standard deviation of rent.] Vacrate is clearly statistically insignificant at 5%, hhsiz is borderline, and percrent is clearly statistically significant at 5%.

(d) (i) The Stata command is test vacrate hhsiz percrent
[Note that if errors are heteroskedastic then we cannot use the usual F-test in terms of sums of squared residuals. For this reason Stata did not give the ANOVA table when the robust option was used in regress. Instead we use $\hat{\beta} \sim \mathcal{N}[0, V]$ gives $W = (R\hat{\beta} - \hat{r})'V^{-1}(R\hat{\beta} - \hat{r}) \sim \chi^2(q)$ under $H_0 : R\beta - r = 0$, where $\hat{V}$ is the heteroskedastic robust estimate. The Stata command test gives an F-version of this $F = W/q$].
(ii) The Stata command is regress rent value, robust

(e) The third equation directly gives the elasticity.
$t = (\hat{\beta} - 1)/s_\beta = (0.5149396 - 1)/0.0208438 = -23.27. |t| > t_{56,.025} \approx 2$.
Very strong rejection of $H_0 : \beta = 1$ against $H_a : \beta \neq 1$.

(f)(i) Plot rent against value along with an OLS regression line and see if variability around the line increases as value increases.
(ii) See whether there is a big difference between heteroskedastic-robust standard errors and standard errors that assume homoskedastic errors.

(g) Run the OLS regression

$$\frac{\text{rent}}{\text{value}} = \beta_1 \frac{1}{\text{value}} + \beta_2 \frac{\text{value}}{\text{value}} + u^*$$
since the error $u^* = u/\text{value} \sim [0, \sigma^2]$ if $u \sim [0, \sigma^2\text{value}^2]$. 

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