# Day 4A
# Asymptotic Theory

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.

*Based on A. Colin Cameron and Pravin K. Trivedi (2009, 2010), Microeconometrics using Stata (MUS), Stata Press.*
*and A. Colin Cameron and Pravin K. Trivedi (2005), Microeconometrics: Methods and Applications (MMA), C.U.P.*

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# 1. Introduction

- Consider sample mean $\bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i$ as an estimator of the population mean $\mu$
  - asymptotic theory gives properties of $\bar{y}$ as $N \rightarrow \infty$.
- Make assumptions about the **data generating process** (dgp)
  - $y_i \sim (\mu, \sigma^2)$ i.i.d. (independent and identically distributed)
  - $\Longrightarrow \bar{y} \sim (\mu, \sigma^2/N)$.

- The distribution of $\bar{y}$ has all mass at the mean of $\mu$ as $N \rightarrow \infty$
  - Intuition: $\mathsf{V}[\bar{y}] = \mathsf{E}[(\bar{y} - \mu)^2] = \sigma^2 / N \rightarrow 0$.
  - Formally $\bar{y}$ **converges in probability** (defined below) to $\mu$
  - This is denoted $\bar{y} \xrightarrow{p} \mu$ or $\operatorname{plim} \bar{y} = \mu$.
  - Then $\bar{y}$ is **consistent** for $\mu$.
  - The proof uses a **law of large numbers** (defined below) for an average.

- Statistical inference based on $\bar{y}$ with $N \longrightarrow \infty$ requires scaling $\bar{y}$ up
  - Use standardized statistic $z = \frac{\bar{y} - \mathrm{E}[\bar{y}]}{\sqrt{\mathrm{V}[\bar{y}]}} = \frac{\bar{y}-\mu}{\sigma/\sqrt{N}}$.
  - $z \sim (0,\ 1)$, so $z$ may have a nondegenerate distribution.
  - A **central limit theorem** (defined below) proves $z \sim \mathcal{N}(0,\ 1)$ as $N \longrightarrow \infty$.
  - Formally $z$ **converges in distribution** ($\xrightarrow{d}$, defined below) to $\mathcal{N}[0, 1]$.
  - Equivalently $\sqrt{N}(\bar{y} - \mu) \xrightarrow{d} \mathcal{N}[0, \sigma^2]$.
  - Note: $\bar{y}$ has been scaled up by the multiple $\sqrt{N}$.

- For simplicity, the formal result $\sqrt{N}(\bar{y}-\mu) \xrightarrow{d} \mathcal{N}[0, \sigma^2]$ is often re-expressed in terms of $\bar{y}$
  - $\bar{y} \overset{a}{\sim} \mathcal{N}[0, \sigma^2/N]$
  - $\overset{a}{\sim}$ means "is asymptotically distributed as"
  - this means $N$ is large enough that the normal is a good approximation
  - but $N$ is not so large that $\sigma^2/N = 0$.

# Outline

# 2. Sequences of random variables

- Recall a sequence of real numbers
  - e.g. $a_N = 2 + \frac{3}{N}$
- What happens as $N \rightarrow \infty$?
  - mathematical convergence (or divergence)
- A sequence of nonstochastic real numbers $\{a_N\}$ *converges* to $a$ if for any $\varepsilon > 0$, there exists $N^* = N^*(\varepsilon)$ such that for all $N > N^*$,

$$|a_N - a| < \varepsilon.$$

  - e.g. $a_N = 2 + 3/N \rightarrow 2$ since $|a_N - a| = |2 + 3/N - 2| = |3/N| < \varepsilon$ for all $N > N^* = 3/\varepsilon$.

- We instead consider a sequence of random variables $b_N$.
  - e.g. $b_N = \frac{1}{N}\sum_{i=1}^N x_i^2$
  - e.g. $b_N = \frac{1}{N}\sum_{i=1}^N x_i u_i$
  - e.g. $b_N = \widehat{\beta} = \left(\frac{1}{N}\sum_{i=1}^N x_i^2\right)^{-1} \frac{1}{N}\sum_{i=1}^N x_i u_i$
- What happens as $N \longrightarrow \infty$?
  - $|b_N - b|$ may exceed $\varepsilon$ due to randomness, so $b_N \nrightarrow b$ exactly
  - instead use convergence in probability.

# 3. Convergence in probability and consistency

- Informal definition: The sequence $\{b_N\}$ *converges in probability* to $b$ if for any $\varepsilon > 0$
$$\lim_{N\to\infty} \Pr[|b_N - b| < \varepsilon] = 1.$$
- Formal definition: A sequence of random variables $\{b_N\}$ *converges in probability* to $b$ if for any $\varepsilon > 0$ and $\delta > 0$, there exists $N^* = N^*(\varepsilon, \delta)$ such that for all $N > N^*$,
$$\Pr[|b_N - b| < \varepsilon] > 1 - \delta.$$
- We write $\text{plim } b_N = b$ or $b_N \xrightarrow{p} b$
  - the limit $b$ may be a constant or a random variable.

# Consistency

- Suppose the sequence $b_N$ is an estimator, say $b_N = \widehat{\boldsymbol{\beta}}$.
  - If $\widehat{\boldsymbol{\beta}} \xrightarrow{p} \boldsymbol{\beta}$, a constant, then we say $\widehat{\boldsymbol{\beta}}$ is *consistent* for $\boldsymbol{\beta}$.
- A simple consistency proof uses *convergence in mean square*
  - that is $\lim_{N \to \infty} \mathrm{E}[(b_N - b)^2] = 0$
  - $\xrightarrow{ms}$ implies convergence in probability.
- Suppose $\widehat{\boldsymbol{\beta}}$ is used to estimate $\boldsymbol{\beta}$
  - $\mathrm{E}[(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})^2] = \mathrm{V}[\widehat{\boldsymbol{\beta}}] + (\mathrm{bias}[\widehat{\boldsymbol{\beta}}])^2$ as MSE = variance + bias$^2$
  - so $\widehat{\boldsymbol{\beta}} \xrightarrow{ms} \boldsymbol{\beta}$ if $\mathrm{V}[\widehat{\boldsymbol{\beta}}] \to 0$ and $\mathrm{bias}[\widehat{\boldsymbol{\beta}}] \to 0$ as $N \to \infty$
  - it follows that $\widehat{\boldsymbol{\beta}} \xrightarrow{p} \boldsymbol{\beta}$ if the variance and bias go to zero as $N \to \infty$.
- We use the weaker convergence in probability as $\widehat{\boldsymbol{\beta}} \xrightarrow{p} \boldsymbol{\beta}$ is possible even if the mean and variance of $\widehat{\boldsymbol{\beta}}$ do not exist.

# 4. Law of large numbers

- Easy way to get probability limit when $b_N$ is an average

$$b_N = \bar{X}_N = \frac{1}{N}\sum_{i=1}^{N} X_i.$$

  - $X_i$ here is general notation for a random variable. e.g. $X_i = x_i u_i$.

- *Weak Law of Large Numbers (WLLN)*:
  Specifies conditions on the individual terms $X_i$ in $\bar{X}_N$ under which

$$(\bar{X}_N - \mathrm{E}[\bar{X}_N]) \xrightarrow{p} 0.$$

- *Khinchine's Theorem (WLLN)*:
  Let $\{X_i\}$ be i.i.d. (independent and identically distributed).
  If and only if $\mathrm{E}[X_i] = \mu$ exists, then $(\bar{X}_N - \mu) \xrightarrow{p} 0$.

- Other LLN's are Kolmogorov and, for i.n.i.d. data, Markov
  - these are given later.
- If a LLN can be applied then

$$\begin{aligned} \operatorname{plim} \bar{X}_N &= \lim \mathrm{E}[\bar{X}_N] && \text{in general} \\ &= \lim N^{-1} \textstyle\sum_{i=1}^{N} \mathrm{E}[X_i] && \text{equivalently} \\ &= \mu && \text{if } X_i \text{ i.i.d.} \end{aligned}$$

# 5. Convergence in distribution

- $b_N$ has (unknown) cumulative distribution function (cdf) $F_N$. Like any other function, $F_N$ may have a limit function.
- *Convergence in Distribution*:
  A sequence of random variables $\{b_N\}$ *converges in distribution* to a random variable $b$ if

$$\lim_{N\to\infty} F_N = F, \text{ where } F \text{ is the c.d.f. of } b$$

  at every continuity point of $F$, where convergence is in the usual mathematical sense.
- We write $b_N \xrightarrow{d} b$, and call $F$ the *limit distribution* of $\{b_N\}$.
- Basically $F_N$ is very complicated and $F$ is simple like $\mathcal{N}[0,1]$.

# 6. Central limit theorems

- Easy way to get limit distribution when $b_N$ is an average $\bar{X}_N$.
- $\bar{X}_N$ has a degenerate limit distribution with all mass at one point since $\bar{X}_N \xrightarrow{p} \lim \mathrm{E}[\bar{X}_N]$ by a LLN.
- So rescale $\bar{X}_N$ to standardized variate

$$b_N = Z_N = \frac{\bar{X}_N - \mathrm{E}[\bar{X}_N]}{\sqrt{\mathrm{V}[\bar{X}_N]}} \sim [0,\ 1].$$

- *Central Limit Theorem (CLT)*:
  A CLT specifies the conditions on the individual terms $X_i$ in $\bar{X}_N$ under which

$$Z_N \xrightarrow{d} \mathcal{N}[0,1].$$

- *Lindeberg-Levy CLT:*
  Let $\{X_i\}$ be i.i.d. with $\mathrm{E}[X_i] = \mu$ and $\mathrm{V}[X_i] = \sigma^2$.
  Then $Z_N = \sqrt{N}(\bar{X}_N - \mu)/\sigma \xrightarrow{d} \mathcal{N}[0,1]$.

- Note that

$$
\begin{aligned}
Z_N &= (\bar{X}_N - \mathrm{E}[\bar{X}_N]) / \sqrt{\mathrm{V}[\bar{X}_N]} && \text{in general} \\
&= \textstyle\sum_{i=1}^N (X_i - \mathrm{E}[X_i]) / \sqrt{\textstyle\sum_{i=1}^N \mathrm{V}[X_i]} && \text{if } X_i \text{ independent over } i \\
&= \sqrt{N}(\bar{X}_N - \mu)/\sigma && \text{if } X_i \text{ i.i.d.}
\end{aligned}
$$

- The last expression can be rewritten as

$$
\frac{\bar{X}_N - \mu}{\sigma/\sqrt{N}} \xrightarrow{d} \mathcal{N}[0,1].
$$

- It follows that $\sqrt{N}(\bar{X}_N - \mu) \xrightarrow{d} \mathcal{N}[0, \sigma^2]$.
- More generally we often find $\sqrt{N}(\widehat{\beta} - \beta) \xrightarrow{d} \mathcal{N}[0, V]$.
  - Scale consistent $\widehat{\beta}$ up by $\sqrt{N}$ to get a limit distribution.

# Multivariate central limit theorem

- Consider vector $\overline{\mathbf{X}}_N$ with mean $\boldsymbol{\mu}_N$ and variance $\mathbf{V}_N$

$$\overline{\mathbf{X}}_N \sim [\boldsymbol{\mu}_N,\ \mathbf{V}_N].$$

- Rescale $\overline{\mathbf{X}}_N$ to standardized variate

$$\mathbf{Z}_N = \mathbf{V}_N^{-1/2}(\overline{\mathbf{X}}_N - \boldsymbol{\mu}_N) \sim [\mathbf{0},\ \mathbf{I}].$$

- *Central Limit Theorem (CLT)*:
  A CLT specifies the conditions on the individual terms $\mathbf{X}_i$ in $\overline{\mathbf{X}}_N$ under which

$$\mathbf{Z}_N \xrightarrow{d} \mathcal{N}[\mathbf{0},\ \mathbf{I}].$$

- Often $\lim N\mathbf{V}_N$ is finite nonzero
  - for example if $\mathbf{X}_i \sim (\boldsymbol{\mu}, \Sigma)$ then $\mathbf{V}_N = \mathrm{V}[\overline{\mathbf{X}}_N] = N^{-1}\Sigma$, so $N\mathbf{V}_N = \Sigma$.
- Then $\mathbf{V}_N^{-1/2}(\overline{\mathbf{X}}_N - \boldsymbol{\mu}_N) \xrightarrow{d} \mathcal{N}[\mathbf{0}, \mathbf{I}]$ implies

$$\sqrt{N}(\overline{\mathbf{X}}_N - \boldsymbol{\mu}_N) \xrightarrow{d} \mathcal{N}[\mathbf{0},\ \lim N^{-1}\mathbf{V}_N]$$

  - scaling the average $\overline{\mathbf{X}}_N$ by a multiple $\sqrt{N}$ gives a limit distribution with a finite nonzero variance.

# 7. Some Key Results

- *Probability Continuity and Continuous Mapping Theorems*
  Let $\mathbf{b}_N$ be a vector of random variables, and $g(\cdot)$ be a continuous real-valued function. Then

$$\begin{array}{rcl} \mathbf{b}_N \xrightarrow{p} \mathbf{b}, \text{ a constant} & \Rightarrow g(\mathbf{b}_N) \xrightarrow{p} g(\mathbf{b}) & \text{Probability Continuity} \\ \mathbf{b}_N \xrightarrow{d} \mathbf{b} & \Rightarrow g(\mathbf{b}_N) \xrightarrow{d} g(\mathbf{b}) & \text{Continuous Mapping} \end{array}$$

- *Transformation Theorem:*
  If $a_N \xrightarrow{d} a$ (a random variable) and $b_N \xrightarrow{p} b$ (a constant), then

$$\begin{array}{ll} (i) & a_N + b_N \xrightarrow{d} a + b \\ (ii) & a_N b_N \xrightarrow{d} ab \\ (iii) & a_N / b_N \xrightarrow{d} a/b, \quad \text{provided } \Pr[b = 0] = 0. \end{array}$$

  - We use especially a matrix version of (ii).

- *Product Limit Normal Rule:*
  For vector $\mathbf{a}_N$ and matrix $\mathbf{H}_N$, if

$$\begin{aligned} \mathbf{a}_N &\xrightarrow{d} \mathcal{N}[\boldsymbol{\mu}, \mathbf{A}] \\ \mathbf{H}_N &\xrightarrow{p} \mathbf{H}, \quad \text{where } \mathbf{H} \text{ is positive definite} \end{aligned}$$

  then

$$\mathbf{H}_N \mathbf{a}_N \xrightarrow{d} \mathcal{N}[\mathbf{H}\boldsymbol{\mu}, \mathbf{HAH}'].$$

- Leading example is OLS:

$$\begin{aligned} \sqrt{N}(\widehat{\beta} - \beta_0) &= (\frac{1}{N}\mathbf{X}'\mathbf{X})^{-1} \times \frac{1}{\sqrt{N}}(\mathbf{X}'\mathbf{u}) \\ &\xrightarrow{d} \mathcal{N}[\mathbf{A}^{-1} \times \mathbf{0}, \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1\prime}]. \end{aligned}$$

# 8. Simulations for LLN and CLT

Uniform on $(0, 1)$ has mean 0 and variance $1/12$.
Sample average of $N$ uniforms has mean 0 and variance $(1/12)/N$.

```
. * Draw from uniform with population mean 0.5
. * Demonstrate LLN by finding average for a very large sample
. * Demonstrate CLT by simulating to obtain many averages
.
. * Small sample: sample mean differs from population mean
. set obs 30
obs was 0, now 30

. set seed 10101

. quietly generate x = runiform()

. mean x

Mean estimation                    Number of obs    =      30

--------------------------------------------------------------
             |       Mean   Std. Err.     [95% Conf. Interval]
-------------+------------------------------------------------
           x |   .5459987   .0511899      .4413036    .6506939
--------------------------------------------------------------
```

For $N = 30$: $\bar{x} = 0.546$ differs appreciably from $\mu = 0.500$.

```
. * Consistency: Large sample: sample mean is very close to population mean
. clear all

. set obs 100000
obs was 0, now 100000

. set seed 10101

. quietly generate x = runiform()

. mean x

Mean estimation                    Number of obs    =   100000
```

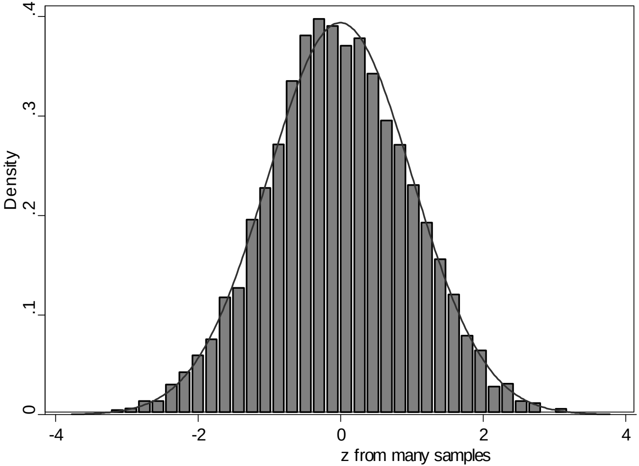
| | Mean | Std. Err. | [95% Conf. Interval] | |
|---|---|---|---|---|
| x | .4988239 | .0009133 | .4970339 | .5006138 |

For $N = 100,000$: $\bar{x} = 0.499$ is very close to $\mu = 0.500$.

```
. * Central limit theorem
. * Write program to obtain sample mean for one sample of size numobs (= 30)
. program onesample, rclass
  1.      args numobs
  2.      drop _all
  3.      quietly set obs `numobs'
  4.      generate x = runiform()
  5.      summarize x
  6.      return scalar meanforonesample = r(mean)
  7. end

. * Run this program 10,000 times to get 10,000 sample means
. quietly simulate xbar = r(meanforonesample), seed(10101) reps(10000) nodots: ///
>    onesample 30

. * Summarize the 10,000 sample means and draw histogram
. summarize xbar

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
        xbar |     10000    .4995835    .0533809   .3008736   .6990562

. histogram xbar, normal xtitle("xbar from many samples")
(bin=40, start=.30087364, width=.00995456)
```

For $S = 10,000$ simulations each with sample size $N = 30$
$\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{10000}$ has distribution with mean 0.4996 close to $\mu = 0.500$ and standard deviation 0.0534, close to $\sigma/\sqrt{N} = \sqrt{1/12}/\sqrt{30} = 0.0527$.

$z = (\bar{x} - \mu)/(\sigma/\sqrt{N}) = (\bar{x} - 0.5)/(\sqrt{1/12}/\sqrt{30}) = (\bar{x} - 0.5)/0.0527.$
Histogram and kernel density estimate for $z_1, z_2, \ldots, z_{10000}$.




This is standard normal, as predicted by the CLT.

# 9. Appendix: Some Further Asymptotic Results

- Alternative modes of convergence of $b_N$ to $b$
  - *Mean square*: $\lim_{N\to\infty} \mathrm{E}[(b_N - b)^2] = 0$.
  - *Chebychev's inequality:* $\Pr[(Z-\mu)^2 > k] \leq \sigma^2/k$, for any $k > 0$.
  - *Almost sure:* $\Pr\{\omega | \lim b_N(\omega) = b(\omega)\} = 1$.
  - These imply convergence in probability.
- Consequences:
  - $b_N \xrightarrow{p} b$ implies $b_N \xrightarrow{d} b$.
  - The reverse is generally not true, unless $b$ is a constant.
  - For vector r.v.'s define $F_N$ and $F$ to be cdf's of vectors $\mathbf{b}_N$ and $\mathbf{b}$.

- *Strong Law of Large Numbers (LLN):*
  - The convergence is instead almost surely ($\xrightarrow{as}$).
- *Kolmogorov SLLN*:
  Let $\{X_i\}$ be i.i.d. If and only if $\mathrm{E}[X_i] = \mu$ exists and $\mathrm{E}[|X_i|] < \infty$, then $(\bar{X}_N - \mathrm{E}[\bar{X}_N]) \xrightarrow{as} 0$.
  - Compared to Khinchine's Theorem $\xrightarrow{as}$ requires $\mathrm{E}[|X_i|] < \infty$.
- *Markov SLLN*:
  Let $\{X_i\}$ be i.n.i.d. with $\mathrm{E}[X_i] = \mu_i$.
  If $\sum_{i=1}^{\infty} \left(\mathrm{E}[|X_i - \mu_i|^{1+\delta}]/i^{1+\delta}\right) < \infty$, for some $\delta > 0$, then $(\bar{X}_N - \mathrm{E}[\bar{X}_N]) \xrightarrow{as} 0$.
  - Relaxes i.i.d. assumption at expense of requiring existence of $(1+\delta)^{th}$ absolute moment.
  - Easiest to set $\delta = 1$, so need variance.

- *Liapounov CLT:*
  Let $\{X_i\}$ be independent with $\mathsf{E}[X_i] = \mu_i$ and $\mathsf{V}[X_i] = \sigma_i^2$ .
  If $\lim \left( \sum_{i=1}^N \mathsf{E}[|X_i - \mu_i|^{2+\delta}] \right) / \left( \sum_{i=1}^N \sigma_i^2 \right)^{(2+\delta)/2} = 0$, for some choice of $\delta > 0$, then $Z_N \xrightarrow{d} \mathcal{N}[0,1]$.
  - The Liapounov CLT relaxes i.i.d. assumption but needs existence of $(2+\delta)^{th}$ absolute moment.
- *Cramer-Wold Device:*
  If $\boldsymbol{\lambda}'\mathbf{b}_N \xrightarrow{d} \mathcal{N}[\ ,\ ]$ for all $\boldsymbol{\lambda} \neq \mathbf{0}$ then $\mathbf{b}_N \xrightarrow{d} \mathcal{N}[\ ,\ ]$.
  - So prove a multivariate CLT by proving a scalar CLT for $\boldsymbol{\lambda}'\mathbf{b}_N$.

# 9. Appendix: Sampling schemes

- Simple Random Sampling (SRS)
  - Randomly draw $(y_i, x_i)$ from the population with equal probabilities.
  - Then $x_i$ i.i.d. So $x_i u_i$ i.i.d. (if errors $u_i$ are i.i.d.), and $x_i^2$ i.i.d.
  - Can use Khinchine's LLN and Lindeberg-Levy CLT.
- Fixed regressors
  - Experiment where $x_i$ are fixed and we observe the resulting random $y_i$.
  - Then $x_i$ fixed, $u_i$ i.i.d. implies $x_i u_i$ i.n.i.d. and $x_i^2$ nonstochastic.
  - Need to use Markov LLN and Liapounov CLT.
- Exogenous Stratified Sampling
  - Oversample some values of $x$ and undersample others.
  - Then $x_i$ i.n.i.d., so $x_i u_i$ i.n.i.d. and $x_i^2$ i.n.i.d.
  - Need to use Markov LLN and Liapounov CLT.

- These three different sampling schemes ultimately lead to similar asymptotic results.
- Microeconometrics often use survey data obtained by stratified sampling.
- The simplest results assume simple random sampling.
- Big problems arise if the stratified sampling is Instead Endogenous Stratified Sampling
  - Oversample some values of $y$ and undersample others.
  - Estimators can be inconsistent.
  - Leading examples are Tobit and selection models.

# 9. Appendix: OLS under simple random sampling

- Scalar regressor: $y_i = \beta x_i + u_i$.
- SRS: $(x_i, y_i)$ i.i.d. with $x_i$ i.i.d. with mean $\mu_x$ & $u_i$ i.i.d. with mean 0.
  - **1.** As $x_i u_i$ are i.i.d. apply Khinchine's Theorem. Then $N^{-1}\sum_i x_i u_i \xrightarrow{p} \mathsf{E}[xu] = \mathsf{E}[x] \times \mathsf{E}[u] = 0$.
  - **2.** As $x_i^2$ are i.i.d. apply Khinchine's Theorem. Then $N^{-1}\sum_i x_i^2 \xrightarrow{p} \mathsf{E}[x^2]$ which we assume exists.
  - **3.** The probability limit is obtained by combining

$$\begin{aligned}
\operatorname{plim}\widehat{\beta} &= \beta + \operatorname{plim}\left(\frac{\frac{1}{N}\sum_{i=1}^{N} x_i u_i}{\frac{1}{N}\sum_{i=1}^{N} x_i^2}\right) \\
&= \beta + \frac{\operatorname{plim}\frac{1}{N}\sum_{i=1}^{N} x_i u_i}{\operatorname{plim}\frac{1}{N}\sum_{i=1}^{N} x_i^2} \\
&= \beta + \frac{0}{\mathsf{E}[x^2]} = \beta,
\end{aligned}$$

using probability limit continuity ($\operatorname{plim}[a_N/b_N] = a/b$ if $b \neq 0$).

- SRS: assume $x_i$ i.i.d. with mean $\mu_x$ and second moment $\mathrm{E}[x^2]$ and assume $u_i$ i.i.d. with mean 0 and variance $\sigma^2$.
- Then $x_i u_i$ are i.i.d. with mean $\mathrm{E}[xu] = \mathrm{E}[x] \times \mathrm{E}[u] = 0$, and $\mathrm{V}[xu] = \mathrm{E}[(xu-0)^2] = \mathrm{E}[x^2u^2] = \mathrm{E}[x^2]\mathrm{E}[u^2] = \sigma^2\mathrm{E}[x^2]$.
  - **1.** Lindeberg-Levy CLT for $N^{-1}\sum_{i=1}^N x_i u_i$ yields
$$\sqrt{N}\left(\frac{N^{-1}\sum_{i=1}^N x_i u_i - 0}{\sqrt{\sigma^2\mathrm{E}[x^2]}}\right) = \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^N x_i u_i}{\sqrt{\sigma^2\mathrm{E}[x^2]}} \xrightarrow{d} \mathcal{N}[0,1].$$
  - **2.** Convert to $\frac{1}{\sqrt{N}}\sum_{i=1}^N x_i u_i$
$$\begin{aligned} \frac{1}{\sqrt{N}}\sum_{i=1}^N x_i u_i &= \sqrt{\sigma^2\mathrm{E}[x^2]} \times \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^N x_i u_i}{\sqrt{\sigma^2\mathrm{E}[x^2]}} \\ &\xrightarrow{d} \sqrt{\sigma^2\mathrm{E}[x^2]} \times \mathcal{N}[0,1] \\ &\xrightarrow{d} \mathcal{N}[0, \sigma^2\mathrm{E}[x^2]] \end{aligned}$$
    using product limit normal rule.

- 3. The limit distribution is obtained by combining

$$\begin{aligned}
\sqrt{N}(\widehat{\beta}-\beta) &= \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N} x_i u_i}{\frac{1}{N}\sum_{i=1}^{N} x_i^2} \\
&\xrightarrow{d} \frac{\mathcal{N}\left[0,\sigma^2 \mathrm{E}[x^2]\right]}{\mathrm{plim}\ \frac{1}{N}\sum_{i=1}^{N} x_i^2} \\
&\xrightarrow{d} \frac{\mathcal{N}\left[0,\sigma^2 \mathrm{E}[x^2]\right]}{\mathrm{E}[x^2]} \\
&\xrightarrow{d} \mathcal{N}\left[0,\sigma^2\left(\mathrm{E}[x^2]\right)^{-1}\right]
\end{aligned}$$

using $\mathrm{plim}\, N^{-1}\sum_{i=1}^{N} x_i^2 = \mathrm{E}[x^2]$ from consistency proof and the product normal limit rule (or $a_N \times b_N \xrightarrow{d} a \times b$ if $a_N \xrightarrow{d} a$ and $b_N \xrightarrow{p} b$).