C2. Health Insurance: Risk Pooling

Health insurance works by pooling individuals together to reduce the variability of potential losses (or risk) faced by an insurance company. The example assumes knowledge of expected value and standard deviation for a random variable and the average of random variables. A review of these concepts is given at the end for completeness but is not essential.

C2.1 Losses for an Individual

Individuals face great variability in their health losses. A year of good health costs little. A year of bad health can be very costly. The possible outcomes for a given individual are summarized using the expected value and the standard deviation.

Example

Suppose a heart attack may occur with probability 0.01 and would cost $50,000 to treat.
Let X denote health costs due to a heart attack.
Then X = $50,000 with probability 0.01
= $0 with probability 0.99.

Mean for an Individual

The expected value of any random variable is the sum of its possible values weighted by the probability of each value:

$$\begin{aligned}\mu &= E[X]\\ &= \Pr[X=\$50{,}000] \times 50{,}000 + \Pr[X=\$0] \times \$0\\ &= 0.01\times\$50{,}000 + 0.99 \times\$0\\ &= \$500.\end{aligned}$$

Standard Deviation for an Individual

To compute the standard deviation we first calculate the variance:

$$\begin{aligned}\sigma^2 &= V[X]\\ &= \Pr[X=\$50{,}000] \times (\$50{,}000 - \$500)^2 + \Pr[X=\$0] \times (\$0 - \$500)^2\\ &= 0.01\times (\$49{,}500)^2 + 0.99 \times (\$500)^2\\ &= 24{,}502{,}500 + 247{,}500 \quad \text{in units of } (\$)^2\\ &= 24{,}750{,}000 .\end{aligned}$$

Then the standard deviation (which unlike the variance is measured in the same units as X)

$$\begin{aligned}\sigma &= S.D.[X]\\ &= (V[X])^{1/2}\\ &= (24{,}750{,}000)^{1/2}\\ &= \$4{,}975\end{aligned}$$

Here the standard deviation is relatively large, being about ten times the expected value.

C2.2 Losses for Health Insurance Company

Health insurance pools a number of individuals together. The insurance company's costs are determined by the average level of annual claims per policy. The insurance company is concerned with both the expected value of average annual claims and the variability (the standard deviation) of average annual claims.

Insurance works because the variability in the average annual claims for 100 policies, say, is much less than the variability in the annual claim on any single policy.
Insurance companies are willing to sell insurance as pooling many people together reduces the variability of their possible losses. And individuals are willing to buy insurance as with insurance the variability of their potential losses is greatly reduced.

A fundamental result is that the variability, as measured by the standard deviation, faced by insurance companies decreases by the square root of the number of people insured.

A second fundamental result is that, for a large insurance pool, average annual claims will be normally distributed. If every person in the pool faces the same distribution of possible outcomes, this normal distribution has mean equal to the expected value of claims for any individual and standard deviation equal to the individual standard deviation divided by the square root of the number of people insured.

Example
We consider 10,000 people for each of whom a heart attack may occur with probability 0.01 and would cost $50,000 to treat.
As shown earlier μ = E[X] = \$500 and σ = E[X] = \$4,975.

Let $\overline{X}$ denote average loss due to heart attack for sample of size 10,000.

Mean of the Average
The expected value of $\overline{X}$ is
$E[\overline{X}] = \mu = \$500$

Standard deviation of the Average
The standard deviation of $\overline{X}$ is
$S.D.[\overline{X}] = \sigma / n^{1/2} = \$4,975 / 10,000^{1/2} = \$49.75$

Distribution of the Average
The sample average $\overline{X}$ is normally distributed with mean \$500 and standard deviation \$49.75.

From standard normal tables it can be shown that for 10,000 insured we can be
    67% sure that average claims will be within \$49.75 of \$500
    95% sure that average claims will be within $2 \times \$49.75 = \$99.50$ of \$500

If instead only 100 are insured we can only be
    67% sure that average claims will be within \$497.50 of \$500.

Normal Distribution

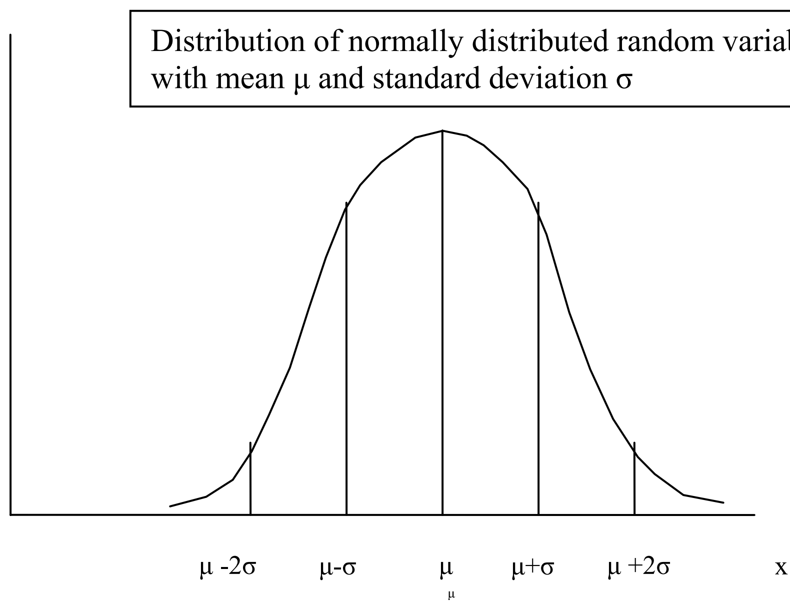
The normal distribution is a bell-shaped curve with formula

$$f(x) = (1/2\pi\sigma^2)^{1/2} \exp(-(x-\mu)^2/2\sigma^2)$$

where $\mu = E[x]$ is the mean of the random variable x

and $\sigma = \text{Sqrt}(V[x])$ is the standard deviation of the random variable x

The normal distribution has the following bell-shape.

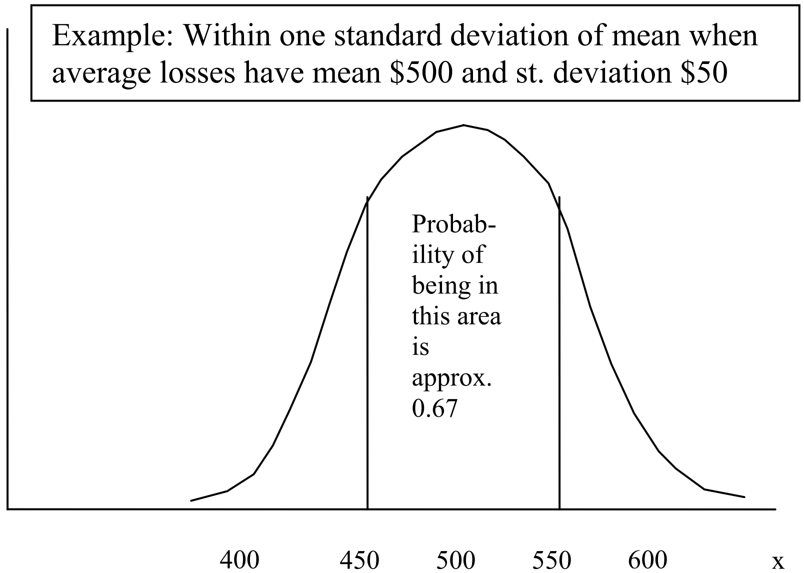


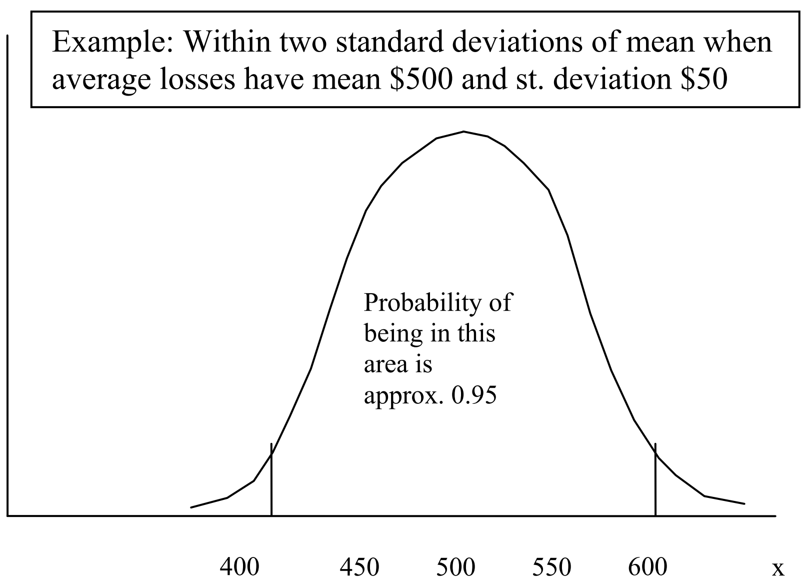
The total area under the curve equals one.

The area under the curve between any two points gives the probability of being between those points.

The probability of being within one standard deviation of the mean is approximately 0.67.

The probability of being within two standard deviations of the mean is approximately 0.95.


Example: Within one standard deviation of mean when average losses have mean $500 and st. deviation $50
Probability of being in this area is approx. 0.67
400
450
500
550
600
x


Example: Within two standard deviations of mean when average losses have mean $500 and st. deviation $50
Probability of being in this area is approx. 0.95
400
450
500
550
600
x

C2.3 More on mean and standard deviation of the Average

The following uses simple numbers to demonstrate that the average $\overline{X}$ has mean $E[\overline{X}] = \mu$ and standard deviation $S.D.[\overline{X}] = \sigma / n^{1/2}$.
i.e. average keeps the same mean but reduces the standard deviation.

Suppose $X = 1$ with probability 0.5
$= 3$ with probability 0.5.

Then
$\mu = E[X] = 0.5 \times 1 + 0.5 \times 3 = 2.$
$\sigma^2 = V[X] = 0.5 \times (1 - 2)^2 + 0.5 \times (3 - 2)^2 = 0.5 \times 1 + 0.5 \times 1 = 1$
$\sigma = S.D.[X] = (V[X])^{1/2} = (1)^{1/2} = 1.$

Suppose we average two such people.
i.e. X_1 is the outcome for the first person and takes values 1 or 3 each with probability 0.5,
and X_2 is the outcome for the second person and takes values 1 or 3 each with probability 0.5.
Then $\overline{X} = (X_1 + X_2)/2$.

According to our theory $\overline{X}$ has
mean $\mu = 1$ and
standard deviation $= \sigma / n^{1/2} = 1 / 2^{1/2} = 0.5^{1/2} =$ **0.707**.

Let's check that from first principles.

There are four possible outcomes for X_1 and X_2.
These are $X_1 = 1$ and $X_2 = 1$ with probability $0.5 \times 0.5 = 0.25$, in which case $\overline{X} = (1 + 1)/2 = 1$
These are $X_1 = 1$ and $X_2 = 3$ with probability $0.5 \times 0.5 = 0.25$, in which case $\overline{X} = (1 + 3)/2 = 2$
These are $X_1 = 3$ and $X_2 = 1$ with probability $0.5 \times 0.5 = 0.25$, in which case $\overline{X} = (3 + 1)/2 = 2$
These are $X_1 = 3$ and $X_2 = 3$ with probability $0.5 \times 0.5 = 0.25$, in which case $\overline{X} = (3 + 3)/2 = 3$

Thus the random variable $\overline{X} = (X_1 + X_2)/2$ takes possible values
$\overline{X} = 1$ with probability 0.25
$\overline{X} = 2$ with probability 0.5
$\overline{X} = 3$ with probability 0.25
so the mean of $\overline{X}$ is
$E[\overline{X}] = 0.25 \times 1 + 0.5 \times 2 + 0.25 \times 3 = 2$
And the variance of $\overline{X}$ is
$Var[\overline{X}] = 0.25 \times (1 - 2)^2 + 0.5 \times (2 - 2)^2 + 0.25 \times (3 - 2)^2 = 0.25 \times 1 + 0.25 \times 1 = 0.5$
And the standard deviation of $\overline{X}$ is
$S.D.[\overline{X}] = (V[\overline{X}])^{1/2} = (0.5)^{1/2} =$ **0.707**.

So the original variance is divided by n = 2 and the standard deviation by the square root of this, as claimed by the general formula.

C2.4 Addendum: Statistics Review of the Mean and Variance of the Sample Mean

Random Variable

X denotes the random variable of interest (for simplicity a discrete random variable).
X takes several different values with different probabilities. A typical value is denoted x.
Pr[X = x] denotes the probability that X takes value x.

Example:
X = loss due to heart attack.
X = \$50,000 with probability 0.01
= \$0 with probability 0.99.

Mean of X

The **mean** is a measure of where the range of values that X may take is centered.
(Other alternative measures are the median and the mode).
The mean is the weighted average of the different values X takes, where the weights are the probability of each particular value occurring.

$\mu = E[X] = \Sigma_x \Pr[X{=}x] \times x$ where Σ_x denotes the sum over the different possible values x

Example:
$E[X] = \Pr[X{=}\$50{,}000] \times 50{,}000 + \Pr[X{=}\$0] \times \$0 = 0.01 \times \$50{,}000 + 0.99 \times \$0 = \$500.$

Variance and standard deviation of X

These are measures of the variation of the range of values that X may take.
(Other alternative measures are the actual range and the inter-quartile range).
The **variance** is the weighted average of the squared difference between each value X takes and the mean value of X, where the weights are the probability of each particular value occurring.
The **standard deviation** is the square root of the variance. It is measured in the same units as X, whereas due to squaring the variance is measured in units of X-squared.

$\sigma^2 = V[X] = E[(X - E[X])^2] = \Sigma_x \Pr[X{=}x] \times (x - E[X])^2$ where Σ_x denotes the sum over x

$\sigma = S.D.[X] = (V[X])^{1/2}$

Example:
$$\begin{aligned} V[X] &= \Pr[X{=}\$50{,}000] \times (\$50{,}000 - \$500)^2 + \Pr[X{=}\$0] \times (\$0 - \$500)^2 \\ &= 0.01 \times (\$49{,}500)^2 + 0.99 \times (\$500)^2 \\ &= 24{,}502{,}500 + 247{,}500 \quad \text{in units of } (\$)^2 \\ &= 24{,}750{,}000\ . \end{aligned}$$

$S.D.[X] = (V[X])^{1/2} = (24{,}750{,}000)^{1/2} = \$4{,}975.$

In this example standard deviation is very large relative to the mean so there is large relative risk.

C2.4 Addendum: Statistics Review (continued)

Sample
A random sample of size n draws randomly n different outcomes of the random variable X.
The sample values are denoted $x_1, x_2, \ldots, x_n$.
Common statistics to explain the sample are the sample mean and sample standard deviation.

Sample Mean
$\overline{X}$ denotes the sample mean.
It is the average of the n sample values denoted $x_1, x_2, \ldots, x_n$.
$\overline{X} = (X_1 + X_2 + \ldots + X_n)/n$

The sample mean is a random variable, since different samples will give different values for $\overline{X}$. Like other random variables, $\overline{X}$ has mean, variance and distribution.

Mean of $\overline{X}$
$$
\begin{aligned}
E[\overline{X}] &= E[\,(X_1 + X_2 + \ldots + X_n) / n\,] \\
&= (E[X_1] + E[X_2] + \ldots + E[X_n]\,) / n \\
&= (\mu + \mu + \ldots + \mu\,) / n \\
&= \mu
\end{aligned}
$$

Variance and standard deviation of $\overline{X}$
$$
\begin{aligned}
V[\overline{X}] &= E[(\overline{X} - E[\overline{X}])^2\,] \\
&= E[(\,(\,(X_1 + X_2 + \ldots + X_n) / n) - \mu)^2\,] \\
&= E[(\,(\,(X_1 - \mu) + (X_2 - \mu) + \ldots + (X_n - \mu)\,) / n)^2\,] \\
&= E[\,(X_1 - \mu)^2 + (X_2 - \mu)^2 + \ldots + (X_n - \mu)^2\,] / n^2 \\
&= (\sigma^2 + \sigma^2 + \ldots + \sigma^2\,) / n^2 \\
&= \sigma^2 / n
\end{aligned}
$$
$S.D.[\overline{X}] = \sigma / n^{1/2}$

Thus **averaging reduces the standard deviation.** Divide by the square root of n.

Example:
Suppose $\overline{X}$ = average loss due to heart attack for sample of size 10,000.
$E[\overline{X}] = \mu = \$500$.
$S.D.[\overline{X}] = \sigma / n^{1/2} = \$4{,}975 / 10{,}000^{1/2} = \$49.75$

Distribution of $\overline{X}$
By the central limit theorem, even if each $x_1, x_2, \ldots, x_n$ is not normally distributed, as the sample size n increases the sample average $\overline{X}$ is normally distributed.

From standard normal tables we know that a random variable lies within one standard deviation of its mean with probability 0.67 and within two standard deviations of its mean with probability 0.95. This implies that

$$\Pr\{\mu - \sigma / n^{1/2} < \overline{X} < \mu - \sigma / n^{1/2}\} = 0.67$$
$$\Pr\{\mu - 2 \text{ x } \sigma / n^{1/2} < \overline{X} < \mu - \sigma / n^{1/2}\} = 0.95.$$

C3. Health Insurance: Risk Aversion

Definition: A person is **risk-averse** if they prefer a certain outcome to a gamble that has expected outcome of the same value.
E.g. prefer \$100 for sure to a 50/50 coin toss with outcome either \$0 or \$200.

A person is **risk-prefer** if instead the gamble is preferred to the certain outcome.
A person is **risk-neutral** if indifferent between the gamble and the certain outcome.
People can be risk-averse, risk-neutral or risk-prefer but it is felt that most people are risk-averse and we consider only this case.

Why buy insurance if the premium is higher than the actuarially fair premium?
Because of **risk-aversion** by the consumer, explained as follows.

C3.1 Model for Insurance

Let x denote a **random outcome**, such as health expenses or income net of health expenses.

Example:

Let x = annual income (in thousands of dollars)

and suppose $x = 50$ with probability 0.5
or $x = 150$ with probability 0.5

Then the expected income is

$$E[x] = 0.5 \times 50 + 0.5 \times 150 = 100.$$

We wish to compare the happiness of a certain 100 to the happiness of the uncertain outcome of either 50 with probability 0.5 or 150 with probability 0.5.

Let U(x) denote the **utility** or happiness that comes from particular values of x.

For concreteness assume the following

Income x	Utility U(x)	Marginal Utility (per 50 change in x)
0	0	0
50	100	100
100	170	70
150	200	30

Utility is increasing in income, as expected, with first derivative $U'(x) > 0$.

This increase in utility declines as income increases, e.g. the first 50 leads to an increase in utility of 100; the next 50 leads to an increase of 70.
This implies that the second derivative $U''(x) < 0$.

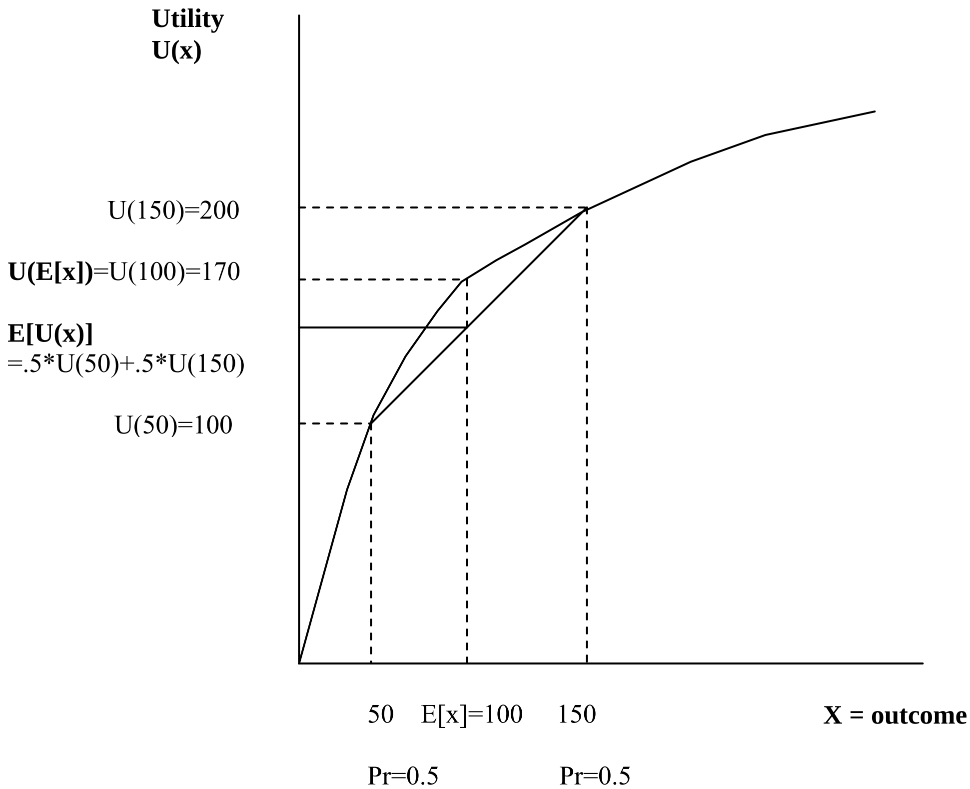
The **expected utility** is

$$E[U(x)] = 0.5 \times U(50) + 0.5 \times U(150) = 0.5 \times 100 + 0.5 \times 200 = 150.$$

For this consumer this is less than the utility of 100 with certainty, since $U(100) = 170 > 150$.

The consumer will therefore consider purchase of insurance against the uncertain outcome, provided the premium is not too high.

C3.2 Risk Averse: Utility of Certain Income > Expected Utility of Uncertain Income



The utility of the expected outcome is 170.
This exceeds the expected utility, the average of the utility of 50 and the utility of 150.
Hence the certain outcome is preferred, as 170 > 150 and the person is risk-averse.

More generally it can be shown that **for a risk-averse consumer the marginal utility of wealth decreases as wealth increases**, giving a utility function that is concave to the origin.

It is clear from the diagram that the **benefit of insurance** against the uncertain outcome is larger for

- **more risk-averse** individuals, i.e. greater curvature U´´ (x) of the utility function.
- **more exposure to risk**, i.e. greater Var[x], the variance of x.

A standard result, used for example in Roger Feldman (1981), "A New Estimate of the Welfare Loss of Excess Health Insurance", *American Economic Review*, Vol.81, pp.297-301, is that the **benefit of risk reduction** equals

$$R \times Var[x]/2,$$

where R= -U´´(x)/ U´(x) is called the coefficient of relative risk aversion.

This benefit is larger for greater risk-aversion (R) and greater exposure to risk (Var[x]).