Machine Learning for Microeconometrics
Part 2: Shrinkage estimators

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Course Outline

- **1:** Variable selection and cross validation
- **Part 2: Shrinkage methods**
  - **ridge, lasso, elastic net**
- **3.** ML for causal inference using lasso
  - OLS with many controls, IV with many instruments
- **4.** Other methods for prediction
  - nonparametric regression, principal components, splines
  - neural networks
  - regression trees, random forests, bagging, boosting
- **5.** More ML for causal inference
  - ATE with heterogeneous effects and many controls.
- **6.** Classification and unsupervised learning
  - classification (categorical y) and unsupervised learning (no y).

1. Introduction

- Consider linear regression model with p potential regressors where p is too large.
- Methods that **reduce the model complexity** are
  - 1. choose a subset of regressors (previous slides)
  - 2. shrink regression coefficients towards zero (these slides)
  - 3. reduce the dimension of the regressors
    - ★ principal components analysis (later slides).
- Linear regression may predict well if include interactions and powers as potential regressors.
- And methods can be adapted to alternative loss functions for estimation.
- **Shrinkage** is also called **regularization**
  - lasso, ridge, elastic net.

Overview

2. Shrinkage: variance-bias trade-offs

- Consider prediction in the regression model

$$y = f(\mathbf{x}) + u \text{ with } E[u] = 0 \text{ and } u \perp \mathbf{x}.$$

- For out-of-estimation-sample point $(y_0, \mathbf{x}_0)$ the true prediction error

$$E[(y_0 - \widehat{f}(\mathbf{x}_0))^2] = Var[\widehat{f}(\mathbf{x}_0)] + \{Bias(\widehat{f}(\mathbf{x}_0))\}^2 + Var(u)$$

- The last term $Var(u)$ is called irreducible error
  - we can do nothing about this.
- So need to **minimize sum of variance and bias-squared**!
  - more flexible models have less bias (good) and more variance (bad).
  - this trade-off is fundamental to machine learning.

Variance-bias trade-off

- Shrinkage is one method that is biased but the bias may lead to lower squared error loss
  - first show this for estimation of a parameter θ
  - then show this for prediction of y.
- The mean squared error of a scalar estimator $\widetilde{\theta}$ is

$$\begin{aligned} \text{MSE}(\widetilde{\theta}) &= E[(\widetilde{\theta} - \theta)^2] \\ &= E[\{(\widetilde{\theta} - E[\widetilde{\theta}]) + (E[\widetilde{\theta}] - \theta)\}^2] \\ &= E[(\widetilde{\theta} - E[\widetilde{\theta}])^2] + (E[\widetilde{\theta}] - \theta)^2 + 2 \times 0 \\ &= Var(\widetilde{\theta}) + Bias^2(\widetilde{\theta}) \end{aligned}$$

  - as the cross product term $2 \times E[(\widetilde{\theta} - E[\widetilde{\theta}])(E[\widetilde{\theta}] - \theta)] = \text{constant} \times E[(\widetilde{\theta} - E[\widetilde{\theta}])] = 0$.

Bias can reduce estimator MSE: a shrinkage example

- Suppose scalar estimator $\widehat{\theta}$ is unbiased for θ with
  - $E[\widehat{\theta}] = \theta$ and $Var[\widehat{\theta}] = v$
  - So MSE$(\widehat{\theta}) = v$.
- Construct the shrinkage estimator $\widetilde{\theta} = a\widehat{\theta}$ where $0 \leq a \leq 1$.
  - Bias: $Bias(\widetilde{\theta}) = E[\widetilde{\theta}] - \theta = a\theta - \theta = (a-1)\theta$.
  - Variance: $Var[\widetilde{\theta}] = Var[a\widehat{\theta}] = a^2 Var(\widehat{\theta}) = a^2 v$
  - So MSE$(\widetilde{\theta}) = Var[\widetilde{\theta}] + Bias^2(\widetilde{\theta}) = a^2 v + (a-1)^2\theta^2$.

Shrinkage example continued

- So we have

$$\begin{array}{lll} \text{Unbiased} & \widehat{\theta} & MSE(\widehat{\theta}) = v \\ \text{Biased} & \widetilde{\theta} = a\widehat{\theta} & \text{MSE}(\widetilde{\theta}) = a^2 v + (a-1)^2\theta^2 \end{array}$$

- Then $\text{MSE}(\widetilde{\theta}) < \text{MSE}[\widehat{\theta}]$ if $\theta^2 < \frac{1+a}{1-a} v$
  - e.g. if $\widetilde{\theta} = 0.9\widehat{\theta}$ then $\widetilde{\theta}$ has lower MSE for $\theta^2 < 19v$!
- We will consider
  - ridge estimator shrinks towards zero
  - LASSO estimator selects and shrinks towards zero.

James-Stein estimator

- Suppose $y_i \sim N(\mu_i, 1)$, $i = 1, ..., n$.
- The MLE is $\widehat{\mu}_i = y_i$ with MSE$(\widehat{\mu}_i) = 1$.
- The James-Stein estimator is $\widetilde{\mu}_i = (1 - c)y_i + c\overline{y}$
  - where $c = \frac{1}{n-3}\sum_{i=1}^{n}(y_i - \overline{y})^2$ and $n \geq 4$
  - this shrinks towards the sample mean $\overline{y}$
- Then MSE$(\widetilde{\mu}_i) < MSE(\widehat{\mu}_i)$ for $n \geq 4$!
- This remarkable 1950's/1960's result was a big surprise
  - an estimator has lower MSE than the maximum likelihood estimator.
- The estimator can be given an empirical Bayes interpretation.

Bias can therefore reduce predictor MSE

- Now consider prediction of $y_0 = \beta x_0 + u$ where $E[u] = 0$
  - using $\widetilde{y}_0 = \widetilde{\beta} x_0$ where treat scalar x_0 as fixed.
- Bias: $Bias(\widetilde{y}_0) = E[x_0\widetilde{\beta}] - \beta x_0 = x_0(E[\widetilde{\beta}] - \beta) = x_0 Bias(\widetilde{\beta})$.
- Variance: $Var[\widetilde{y}_0] = Var[x_0\widetilde{\beta}] = x_0^2 Var(\widetilde{\beta})$.
- The mean squared error in the scalar regressor case is

$$\begin{aligned} \text{MSE}(\widetilde{y}_0) &= Var(\widetilde{y}_0) + Bias^2(\widetilde{y}_0) + Var(u) \\ &= x_0^2 Var(\widetilde{\beta}) + (x_0 Bias(\widetilde{\beta}))^2 + Var(u) \\ &= x_0^2\{Var(\widetilde{\beta}) + Bias^2(\widetilde{\beta})\} + Var(u) \\ &= x_0^2 MSE(\widetilde{\beta}) + Var(u). \end{aligned}$$

- So bias in $\widetilde{\beta}$ that reduces $\text{MSE}(\widetilde{\beta})$ also reduces $\text{MSE}(\widetilde{y}_0)$.

3. Shrinkage Methods

- Shrinkage estimators minimize RSS (residual sum of squares) with a penalty for overfitting the data at hand.
  - this shrinks parameter estimates towards zero.
- The extent of shrinkage is determined by a **tuning parameter**
  - this is determined by cross-validation or penalty such as AIC or BIC.
- **Standardize regressors** as ridge, LASSO and elastic net are not invariant to rescaling of regressors
  - so x_{ij} below is actually $(x_{ij} - \bar{x}_j)/s_j$
  - and demean y_i so below y_i is actually $y_i - \bar{y}$
  - $\mathbf{x}_i$ does not include an intercept nor does data matrix $\mathbf{X}$
  - we can recover intercept β_0 as $\widehat{\beta}_0 = \bar{y}$.
- So work with $y = \mathbf{x}'\boldsymbol{\beta} + \varepsilon = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p + \varepsilon$.

3.1 Ridge Regression

- The simplest form of the **ridge estimator** $\widehat{\boldsymbol{\beta}}_\lambda$ of $\boldsymbol{\beta}$ minimizes

$$Q_\lambda(\boldsymbol{\beta}) = \frac{1}{n}\sum_{i=1}^{n}(y_i - \mathbf{x}_i'\boldsymbol{\beta})^2 + \lambda\sum_{j=1}^{p}\beta_j^2 = RSS + \lambda(||\boldsymbol{\beta}||_2)^2$$

  - where $\lambda \geq 0$ is a tuning parameter to be determined
  - $||\boldsymbol{\beta}||_2 = \sqrt{\sum_{j=1}^{p}\beta_j^2}$ is L2 norm.

- Equivalently the ridge estimator minimizes

$$\frac{1}{n}\sum_{i=1}^{n}(y_i - \mathbf{x}_i'\boldsymbol{\beta})^2 \text{ subject to } \sum_{j=1}^{p}\beta_j^2 \leq s.$$

- The ridge estimator is

$$\widehat{\boldsymbol{\beta}}_\lambda = (\mathbf{X}'\mathbf{X} + n\lambda\mathbf{I})^{-1}\mathbf{X}'\mathbf{y}.$$

- More generally can weight each β_j
  - $Q_\lambda(\boldsymbol{\beta}) = \frac{1}{n}\sum_{i=1}^{n}(y_i - \mathbf{x}_i'\boldsymbol{\beta})^2 + \lambda\sum_{j=1}^{p}\kappa_j\beta_j^2$.

Ridge Derivation

- 1. Objective function includes penalty
  - $Q(\boldsymbol{\beta}) = \frac{1}{n}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) + \lambda\boldsymbol{\beta}'\boldsymbol{\beta}$
  - $\partial Q(\boldsymbol{\beta})/\partial\boldsymbol{\beta} = -\frac{2}{n}\mathbf{X}'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) + 2\lambda\boldsymbol{\beta} = \mathbf{0}$
  - $\Rightarrow \mathbf{X}'\mathbf{X}\boldsymbol{\beta} + \lambda\mathbf{I}\boldsymbol{\beta} = \mathbf{X}'\mathbf{y}$
  - $\Rightarrow \widehat{\boldsymbol{\beta}}_\lambda = (\mathbf{X}'\mathbf{X} + n\lambda\mathbf{I})^{-1}\mathbf{X}'\mathbf{y}$.
- 2. Form Lagrangian (multiplier is λ) from objective function and constraint
  - $Q(\boldsymbol{\beta}) = \frac{1}{n}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})$ and constraint $\boldsymbol{\beta}'\boldsymbol{\beta} \leq s$
  - $L(\boldsymbol{\beta}, \lambda) = \frac{1}{n}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) + \lambda(\boldsymbol{\beta}'\boldsymbol{\beta} - s)$
  - $\partial L(\boldsymbol{\beta}, \lambda)/\partial\boldsymbol{\beta} = -\frac{2}{n}\mathbf{X}'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) + 2\lambda\boldsymbol{\beta} = \mathbf{0}$
  - $\Rightarrow \widehat{\boldsymbol{\beta}}_\lambda = (\mathbf{X}'\mathbf{X} + n\lambda\mathbf{I})^{-1}\mathbf{X}'\mathbf{y}$
  - Here $\lambda = \partial L_{opt}(\boldsymbol{\beta}, \lambda, s)/\partial s$.

Ridge Properties

- $\widehat{\boldsymbol{\beta}}_\lambda \rightarrow \mathbf{0}$ as $\lambda \rightarrow \infty$ and $\widehat{\boldsymbol{\beta}}_\lambda \rightarrow \widehat{\boldsymbol{\beta}}_{OLS}$ as $\lambda \rightarrow 0$.
- Ridge best when many predictors important with coeffs of similar size.
- Ridge best when LS has high variance
  - meaning small changes in training data can lead to large changes in OLS coefficient estimates.
- Algorithms exist to quickly compute $\widehat{\boldsymbol{\beta}}_\lambda$ for many values of λ
  - then choose λ by cross validation or AIC or BIC.
  - with search over a decreasing logarithmic grid in λ.

More on Ridge

- Also called Tikhonov regularization.
- Hoerl & Kennard (1970) proposed ridge as a way to reduce MSE of $\widehat{\boldsymbol{\beta}}$.
- We can write ridge as $\widehat{\boldsymbol{\beta}}_\lambda = (\mathbf{X}'\mathbf{X} + n\lambda\mathbf{I})^{-1}\mathbf{X}'\mathbf{X} \times \widehat{\boldsymbol{\beta}}_{OLS}$
  - so shrinkage of OLS toward zero.
- For scalar regressor and no intercept $\widehat{\beta}_\lambda = a\widehat{\beta}_{OLS}$ where $a = \frac{\sum_i x_i^2}{\sum_i x_i^2 + n\lambda}$
  - like earlier example of $\widetilde{\beta} = a\widehat{\beta}$.
- Ridge is the posterior mean for $\mathbf{y} \sim N(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I})$ with prior $\boldsymbol{\beta} \sim N(0, \gamma^2\mathbf{I})$
  - though γ is a specified prior parameter whereas λ is data-determined.
- Ridge is estimator in model $\mathbf{y} \sim (\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I})$ with stochastic constraints $\boldsymbol{\beta} \sim (0, \gamma^2\mathbf{I})$.

3.2 LASSO (Least Absolute Shrinkage And Selection Operator)

- The **LASSO estimator** $\widehat{\boldsymbol{\beta}}_\lambda$ of $\boldsymbol{\beta}$ minimizes

$$Q_\lambda(\boldsymbol{\beta}) = \frac{1}{n}\sum_{i=1}^{n}(y_i - \mathbf{x}_i'\boldsymbol{\beta})^2 + \lambda\sum_{j=1}^{p}|\beta_j| = RSS + \lambda||\boldsymbol{\beta}||_1$$

  - where $\lambda \geq 0$ is a tuning parameter to be determined
  - $||\boldsymbol{\beta}||_1 = \sum_{j=1}^{p}|\beta_j|$ is L1 norm.

- Equivalently the LASSO estimator minimizes

$$\frac{1}{n}\sum_{i=1}^{n}(y_i - \mathbf{x}_i'\boldsymbol{\beta})^2 \text{ subject to } \sum_{j=1}^{p}|\beta_j| \leq s.$$

- Features
  - best when a few regressors have $\beta_j \neq 0$ and most $\beta_j = 0$
  - leads to a more interpretable model than ridge.

- More generally $Q_\lambda(\boldsymbol{\beta}) = \frac{1}{n}\sum_{i=1}^{n}(y_i - \mathbf{x}_i'\boldsymbol{\beta})^2 + \lambda\sum_{j=1}^{p}\kappa_j|\beta_j|$.

LASSO versus Ridge (key figure from ISL)

- LASSO is likely to set some coefficients to zero.

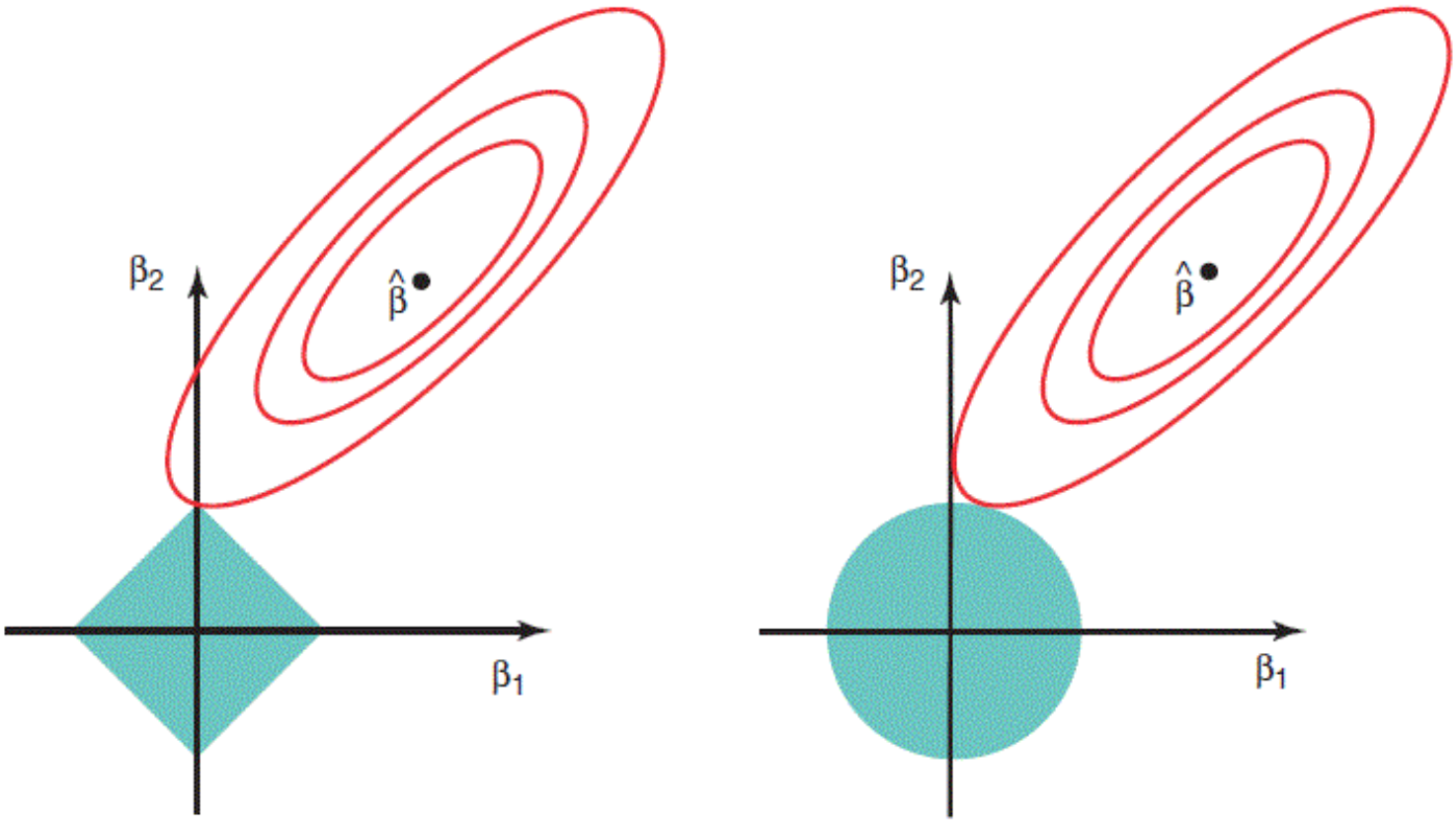


FIGURE 6.7. *Contours of the error and constraint functions for the lasso* (left) *and ridge regression* (right). *The solid blue areas are the constraint regions,* $|\beta_1| + |\beta_2| \leq s$ *and* $\beta_1^2 + \beta_2^2 \leq s$, *while the red ellipses are the contours of the RSS.*

LASSO versus Ridge

- Consider simple case where $n = p$ and $\mathbf{X} = \mathbf{I}_n$ (identity matrix).
- OLS: $\widehat{\boldsymbol{\beta}}^{OLS} = (\mathbf{I}'\mathbf{I})^{-1}\mathbf{I}'\mathbf{y} = \mathbf{y}$ so $\widehat{\beta}_j^{OLS} = y_j$
- Ridge shrinks all $\beta's$ towards zero

$$\begin{aligned} \widehat{\boldsymbol{\beta}}^{R} &= (\mathbf{I}'\mathbf{I} + \lambda\mathbf{I})^{-1}\mathbf{I}'\mathbf{y} = \mathbf{y}/(1+\lambda) \\ \widehat{\beta}_j^{R} &= y_j/(1+\lambda) \end{aligned}$$

- LASSO shrinks some $\beta_j's$ towards 0 and sets others $= 0$

$$\widehat{\beta}_j^{L} = \begin{cases} y_j - \lambda/2 & \text{if } y_j > \lambda/2 \\ y_j + \lambda/2 & \text{if } y_j < -\lambda/2 \\ 0 & \text{if } |y_j| \leq \lambda/2 \end{cases}$$

- Aside: Best subset of size M in this example

$$\widehat{\beta}_j^{BS} = \widehat{\beta}_j \times \mathbf{1}[|\widehat{\beta}_j| \geq |\widehat{\beta}_{(M)}|]$$

where $\widehat{\beta}_{(M)}$ is the M^{th} largest OLS coefficient.

Computation of LASSO estimator

- Most common is a coordinate wise descent algorithm
  - also called a shooting algorithm due to Fu (1998)
  - exploits the special structure in the nondifferentiable part of the LASSO objective function that makes convergence possible.
- The algorithm for given λ (λ is later chosen by CV)
  - denote $\boldsymbol{\beta} = (\beta_j, \boldsymbol{\beta}^{-j})$ and define $S_j(\beta_j, \boldsymbol{\beta}^{-j}) = \partial RSS/\partial \beta_j$
  - start with $\widehat{\boldsymbol{\beta}} = \widehat{\boldsymbol{\beta}}_{OLS}$
  - at step m for each $j = 1, ..., p$ let $S_0 = S_j(0, \widehat{\boldsymbol{\beta}}^{-j})$ and set

$$\widehat{\beta}_j = \begin{cases} \frac{\lambda - S_0}{2\mathbf{x}_j'\mathbf{x}_j} & \text{if } S_0 > \lambda \\ \frac{-\lambda - S_0}{2\mathbf{x}_j'\mathbf{x}_j} & \text{if } S_0 < -\lambda \\ 0 & \text{if } -\lambda < S_0 < \lambda \end{cases}$$

  - form new $\widehat{\boldsymbol{\beta}}_m = [\widehat{\beta}_1 \cdots \widehat{\beta}_p]$ after updating all $\widehat{\beta}_j$.
- Alternatively LASSO is a minor adaptation of least angle regression
  - so estimate using the forward-stagewise algorithm for LAR.

More on LASSO

- LASSO is due to Tibshirani (1999).
- Can weight each β_j differently
  - implemented in Stata `lasso` commands.
- Can specify some variables to be always included.
- The group lasso allows to include regressors as groups (e.g. race dummies as a group)
  - with L groups minimize over $\boldsymbol{\beta}$

$$\frac{1}{n}\sum_{i=1}^{n}\left(y_i-\sum_{l=1}^{L}\mathbf{x}_i'\boldsymbol{\beta}_l\right)^2+\lambda\sum_{l=1}^{L}\sqrt{\rho_l}\left(\sum_{j=1}^{p_l}|\beta_{lj}|\right).$$

- There are other extensions such as adaptive LASSO.
- Giannone, Lenza and Primiceri (2021) find that sparse models (e.g. LASSO) predict poorly in several standard economic applications. Shrinkage (e.g. Ridge) predicts better.

3.3 Elastic net

- Elastic net combines ridge regression and LASSO with objective function

$$Q_{\lambda,\alpha}(\boldsymbol{\beta}) = \frac{1}{n}\sum_{i=1}^{n}(y_i - \mathbf{x}_i'\boldsymbol{\beta})^2 + \lambda\sum_{j=1}^{p}\{\alpha|\beta_j| + (1-\alpha)\beta_j^2\}.$$

  - ridge penalty λ averages correlated variables
  - LASSO penalty α leads to sparsity.
- For elastic net
  - Ridge is special case $\alpha = 0$
  - LASSO is special case $\alpha = 1$.
- K-fold cross validation is used with default $K = 10$
  - `set seed` for replicability.

3.4 Asymptotic Properties of Lasso

- **A model selection method is consistent** if asymptotically it correctly selects the correct model from a selection of candidate models
  - selecting on basis of minimum BIC is consistent.
- **A model selection method is conservative** if asymptotically it always selects a model that nests the correct model
  - selecting a model on the basis of minimum AIC is conservative.
  - Hannes Leeb and Benedikt M. Pötscher (2005), "Model Selection and Inference", *Econometric Theory*, 21-59.
- A statistical model selection and estimation method is said to have an **oracle property** if it leads to consistent model selection and a subsequent estimator that is asymptotically equivalent to the estimator that could be obtained if the true model was known so that model selection was unnecessary.

Asymptotic properties of Lasso

- The LASSO is a consistent model selection procedure
  - but does not have oracle property due to bias.
- The oracle property is an asymptotic property
  - not useful in finite sample settings that economists encounter
    - ★ our models do not fit perfectly
  - and gives rates for a penalty parameters but not finite sample value.
- Lasso estimates have complicated finite sample distribution.
  - cannot perform standard inference on LASSO or post_LASSO
  - instead add some model structure
    - ★ e.g. partial linear model.

3.5 Clustered Data

- Now consider clustered data
  - by clustered data I mean "clustered errors"
  - data are grouped with correlated observations within group and uncorrelated across groups
  - examples are panel data and grouping by independent regions.
- Notation: y_{ig} is outcome for individual i in cluster g, $i = 1, ..., N_g$, $g = 1, ..., G$.
- Here focus on lasso as the machine learning method.

Lasso with Clustered Data

- We want to generalize $Q_\lambda(\boldsymbol{\beta}) = \frac{1}{n}\sum_{i=1}^{n}(y_i - \mathbf{x}_i'\boldsymbol{\beta})^2 + \lambda\sum_{j=1}^{p}|\beta_j|$.
- **Method 1.** With clustered data one can continue with this in which case equal weight is given to each observation

$$Q_\lambda(\boldsymbol{\beta}) = \frac{1}{G}\sum_{g=1}^{G}\sum_{i=1}^{N_g}(y_{ig} - \mathbf{x}_{ig}'\boldsymbol{\beta})^2 + \lambda\sum_{j=1}^{p}|\beta_j|$$

- **Method 2.** Alternatively one can give equal weight to each cluster

$$Q_\lambda(\boldsymbol{\beta}) = \frac{1}{G}\sum_{g=1}^{G}\frac{1}{N_g}\sum_{i=1}^{N_g}(y_{ig} - \mathbf{x}_{ig}'\boldsymbol{\beta})^2 + \lambda\sum_{j=1}^{p}|\beta_j|$$

- Stata 17 option `cluster(`*clustervar*`)` of the `lasso` command does method 2.
- Which is best?
  - If data are independent within cluster then 1.?
  - If data are perfectly correlated within cluster then 2.?
  - And a big difference if cluster sizes are greatly unbalanced.

Key papers

- Peter Bickel, Ya'acov Ritov and Alexandre Tsybakov (2009), "Simultaneous Analysis of Lasso and Dantzig Selector", *The Annals of Statistics*, 1705-1732.
  - In reading this just consider Lasso (ignore Dantzig Selector).
  - lays out the typical assumptions well
    - ★ including sparsity and $y_i = f(\mathbf{z}_i) + u_i$, u_i i.i.d. $N(0, \sigma^2)$
  - lays out finite sample bounds for prediction loss
    - ★ key are assumptions on the eigenvalues of the Gram matrix $\mathbf{X}'\mathbf{X}$
    - ★ similar to restrictions on the correlations among regressors.
- Alex Belloni and Victor Chernozhukov (2013), "Least Squares after Model Selection in High-Dimensional Sparse Models", *Bernoulli*, 521-547.
  - builds on the previous paper
  - harder to read, so read Bickel et al first
  - OLS after LASSO is better than prediction from LASSO estimates.

4. Prediction using LASSO: Stata lasso command

- lasso *model* depvar [(*alwaysvars*)] *othervars*, options
- Model is
  - `linear, logit, probit` or `poisson`
- `folds(#)`
- penalty parameter λ
  - cross validation (`selection(cv)`) sets all $\kappa_j = 1$
  - adaptive cv (`selection(adaptive cv)`) κ_j can vary
  - AIC (`bic`)
  - plug-in (`selection(plugin)`) for non-prediction applications

Postestimation commands

- lasso command focuses on finding λ
- Following commands give more info
  - `lassoknots`
  - `lassoselect`
  - `cvplot`
  - `coefpath`
  - `lassoinfo`
  - `lassocoef`
  - `lassogof`

4.2 LASSO linear regression example

- Generated data example: $n = 40$, $p = 3$.
- Three correlated regressors.
  - $$\begin{bmatrix} x_{1i} \\ x_{2i} \\ x_{3i} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix} \right)$$
- But only x_1 determines y
  - $y = 2 + x_{1i} + u_i$ where $u_i \sim N(0, 3^2)$.
- Same generate data as in part 1 slides.

```
. * Summarize data
. summarize
```

Variable	Obs	Mean	Std. Dev.	Min	Max
x1	40	.3337951	.8986718	-1.099225	2.754746
x2	40	.1257017	.9422221	-2.081086	2.770161
x3	40	.0712341	1.034616	-1.676141	2.931045
y	40	3.107987	3.400129	-3.542646	10.60979

```
. correlate
(obs=40)
```

	x1	x2	x3	y
x1	1.0000			
x2	0.5077	1.0000		
x3	0.4281	0.2786	1.0000	
y	0.4740	0.3370	0.2046	1.0000

Aside: Demeaning data

- Stata commands such as `lasso` do this automatically
  - but for completeness following code demeans.

```
. * Standardize regressors and demean y
. foreach var of varlist x1 x2 x3 {
  2.    qui egen z`var' = std(`var')
  3.    }

. quietly summarize y

. quietly generate ydemeaned = y - r(mean)

. summarize ydemeaned z*
```

Variable	Obs	Mean	Std. Dev.	Min	Max
ydemeaned	40	-1.71e-08	3.400129	-6.650633	7.501798
zx1	40	2.05e-09	1	-1.594598	2.693921
zx2	40	2.79e-10	1	-2.34211	2.80662
zx3	40	2.79e-09	1	-1.688912	2.764129

- The original variables x_1 to x_3 had standard deviations 0.89867, 0.94222 and 1.03462
  - means differ from zero due to single precision rounding error.

Demeaning data better

- Aside: Use double precision

```
. * Standardize regressors and demean y
. foreach var of varlist x1 x2 x3 {
  2.      qui egen double z`var' = std(`var')
  3. }

. qui summarize y

. qui generate double ydemeaned = y - r(mean)

. summarize ydemeaned z*
```

Variable	Obs	Mean	Std. Dev.	Min	Max
ydemeaned	40	-3.33e-17	3.400129	-6.650633	7.501798
zx1	40	2.63e-17	1	-1.594598	2.693921
zx2	40	2.62e-17	1	-2.34211	2.80662
zx3	40	-2.98e-17	1	-1.688912	2.764129

- Stata does internal calculations in double precision but default is to save variables in single precision.

LASSO linear regression example: lasso command

- Apply to generated data example: $n = 40$, $K = 5$, $p = 3$.
  - **set the seed !**
  - First regressor selected when $\lambda = 1.450138$

```
. * Lasso linear using 5-fold cross validation
. lasso linear y x1 x2 x3, selection(cv) folds(5) rseed(10101)

5-fold cross-validation with 100 lambdas ...
Grid value 1:     lambda = 1.591525   no. of nonzero coef. =       0
Folds: 1...5   CVF = 11.85738
Grid value 2:     lambda = 1.450138   no. of nonzero coef. =       1
Folds: 1...5   CVF = 11.60145
Grid value 3:     lambda = 1.321312   no. of nonzero coef. =       1
Folds: 1...5   CVF = 11.2296
Grid value 4:     lambda =  1.20393   no. of nonzero coef. =       1
Folds: 1...5   CVF = 10.87719
Grid value 5:     lambda = 1.096976   no. of nonzero coef. =       1
Folds: 1...5   CVF = 10.60149
Grid value 6:     lambda = .9995238   no. of nonzero coef. =       1
Folds: 1...5   CVF = 10.38463
Grid value 7:     lambda = .9107289   no. of nonzero coef. =       1
Folds: 1...5   CVF = 10.20522
Grid value 8:     lambda = .8298222   no. of nonzero coef. =       1
Folds: 1...5   CVF = 10.05685
```

lasso command (continued)

- Second regressor included when $\lambda = 0.6277301$

```
Grid value 9:     lambda = .7561031   no. of nonzero coef. =       1
Folds: 1...5   CVF = 9.934201
Grid value 10:    lambda =  .688933   no. of nonzero coef. =       1
Folds: 1...5   CVF = 9.829713
Grid value 11:    lambda = .6277301   no. of nonzero coef. =       2
Folds: 1...5   CVF = 9.739804
Grid value 12:    lambda = .5719643   no. of nonzero coef. =       2
Folds: 1...5   CVF = 9.666469
Grid value 13:    lambda = .5211525   no. of nonzero coef. =       2
Folds: 1...5   CVF = 9.606777
Grid value 14:    lambda = .4748548   no. of nonzero coef. =       2
Folds: 1...5   CVF = 9.562824
Grid value 15:    lambda =   .43267   no. of nonzero coef. =       2
Folds: 1...5   CVF = 9.525748
Grid value 16:    lambda = .3942328   no. of nonzero coef. =       2
Folds: 1...5   CVF = 9.493472
Grid value 17:    lambda = .3592102   no. of nonzero coef. =       2
Folds: 1...5   CVF = 9.460115
Grid value 18:    lambda =  .327299   no. of nonzero coef. =       2
Folds: 1...5   CVF = 9.43311
Grid value 19:    lambda = .2982226   no. of nonzero coef. =       2
Folds: 1...5   CVF = 9.411316
```

lasso command (continued)

- Minimum CV of 9.393523 with two regressors.

```
Grid value 20:    lambda = .2717294   no. of nonzero coef. =       2
Folds: 1...5   CVF = 9.393794
Grid value 21:    lambda = .2475897   no. of nonzero coef. =       2
Folds: 1...5   CVF = 9.393523
Grid value 22:    lambda = .2255945   no. of nonzero coef. =       2
Folds: 1...5   CVF = 9.40661
Grid value 23:    lambda = .2055533   no. of nonzero coef. =       2
Folds: 1...5   CVF = 9.420332
Grid value 24:    lambda = .1872925   no. of nonzero coef. =       2
Folds: 1...5   CVF = 9.434326
... cross-validation complete ... minimum found
```

- Default grid search is a decreasing logarithmic grid of 100 values
  - $\lambda_j = \lambda_1 \times 10^{-4(j-1)/99}$, $j = 2, .., 100$
  - $\lambda_1 = 1.591525$ is the smallest value at which no regressors are selected.

lasso command (continued)

- Final results on optimal λ.

```
Lasso linear model                          No. of obs         =         40
                                            No. of covariates  =          3
Selection: Cross-validation                 No. of CV folds    =          5
```

ID	Description	lambda	No. of nonzero coef.	Out-of-sample R-squared	CV mean prediction error
1	first lambda	1.591525	0	-0.0519	11.85738
20	lambda before	.2717294	2	0.1666	9.393794
* 21	selected lambda	.2475897	2	0.1666	9.393523
22	lambda after	.2255945	2	0.1655	9.40661
24	last lambda	.1872925	2	0.1630	9.434326

* lambda selected by cross-validation.

lassoknots command

- Lists values of λ at which variables are added and removed
  - here first x1 and then x2 are added.

```
. * List the values of lambda at which variables are added or removed
. lassoknots
```

ID	lambda	No. of nonzero coef.	CV mean pred. error	Variables (A)dded, (R)emoved, or left (U)nchanged
2	1.450138	1	11.60145	A x1
11	.6277301	2	9.739804	A x2
* 21	.2475897	2	9.393523	U
24	.1872925	2	9.434326	U

```
* lambda selected by cross-validation.
```

cvplot command

- Plot value of CV_5 against λ on log scale
  - simply command `cvplot`
- Plot of how coefficients change with λ
  - command `coefpath`

lassoinfo command

- Provides a summary of the LASSO.

```
. * Provide a summary of the lasso
. lassoinfo

    Estimate:  active
     Command:  lasso
```

Depvar	Model	Selection method	Selection criterion	lambda	No. of selected variables
y	linear	cv	CV min.	.2475897	2

lassocoef command

- Provides three different sets of coefficient estimates
- **1.** Standardized coefficients (default) are those directly from lasso.
- **2.** Penalized coefficients are the preceding ones rescaled so that the standardization of variables is removed
  - so multiply each coefficient by the standard deviation of the corresponding regressor.
  - i.e. can interpret in terms of the original data.
- **3.** Post-selection coefficients are obtained by OLS of `y` on the selected regressors (here `x1` and `x2`).
  - often called post-lasso estimates.

Standardized coefficients for standardized regressors

- We have

```
. * Lasso coefficients for the standardized regressors
. lassocoef, display(coef, standardized)
```

	active
x1	1.206056
x2	.2715635
_cons	0

```
Legend:
  b - base level
  e - empty cell
  o - omitted
```

Unstandardized coefficients for original regressors

- Recall d.g.p. $y = 2 + 1 \times x_1 + 0 \times x_2 + 0 \times x_3 + u$.

```
. *  Lasso coefficients for the unstandardized regressors
. lassocoef, display(coef, penalized) nolegend
```

	active
x1	1.35914
x2	.2918877
_cons	2.617622

Post-selection (post-LASSO)

- OLS on the selected and unstandardized regressors
  - same as `regress y x1 x2`

```
. * Post-selection estimated coefficients for the unstandardized regressors
. lassocoef, display(coef, postselection) nolegend
```

	active
x1	1.544198
x2	.4683922
_cons	2.533663

lassogof command

- Goodness-of-fit
  - as expected post-lasso OLS fits better than lasso in the full sample
  - since OLS minimizes MSE while lasso minimizes MSE plus penalty

```
. * Goodness-of-fit with penalized coefficients and postselection coefficients
. lassogof, penalized

Penalized coefficients
------------------------------------------------------------------------------
             MSE     R-squared           Obs
------------------------------------------------------------------------------
        8.679274        0.2300            40
------------------------------------------------------------------------------

. lassogof, postselection

Postselection coefficients
------------------------------------------------------------------------------
             MSE     R-squared           Obs
------------------------------------------------------------------------------
        8.597958        0.2372            40
------------------------------------------------------------------------------
```

Adaptive Lasso

- Method that usually leads to fewer variables than basic lasso.
  - First lasso as usual with $\kappa_j = 1$ since then $\lambda \sum_{j=1}^{p} \kappa_j |\beta_j| = \lambda \sum_{j=1}^{p} |\beta_j|$.
  - Second exclude x_j with $\widehat{\beta}_j = 0$ and for remainder set $\kappa_j = 1/|\widehat{\beta}_j|^{\delta}$ with default $\delta = 1$.
- Here only `x1` is selected.

```
. * Lasso linear using 5-fold adaptive cross validation
. qui lasso linear y x1 x2 x3, selection(adaptive) folds(5) rseed(10101)

. lassoknots
```

ID	lambda	No. of nonzero coef.	CV mean pred. error	Variables (A)dded, (R)emoved, or left (U)nchanged
26	3.945214	1	11.60145	A x1
* 52	.3512089	1	9.160539	U
57	.2205694	2	9.210699	A x2
95	.0064297	2	9.172378	U

* lambda selected by cross-validation in final adaptive step.

4.5 Elasticnet and Ridge Regression

- Elastic net combines ridge regression and LASSO with objective function

$$Q_{\lambda,\alpha}(\boldsymbol{\beta}) = \sum_{i=1}^{n}(y_i - \mathbf{x}_i'\boldsymbol{\beta})^2 + \lambda \sum_{j=1}^{p}\{\alpha|\beta_j| + (1-\alpha)\beta_j^2\}.$$

  - ridge penalty λ averages correlated variables
  - LASSO penalty α leads to sparsity.
- For elastic net
  - Ridge is special case $\alpha = 0$
  - LASSO is special case $\alpha = 1$.

Ridge Regression

- Standardized ridge (OLS estimates were 1.555, 0.471 and -0.026.)

```
. * Ridge estimation using the elasticnet command and selected results
. qui elasticnet linear y x1 x2 x3, alpha(0) rseed(10101) folds(5)

. lassoknots
```

alpha	ID	lambda	No. of nonzero coef.	CV mean pred. error	Variables (A)dded, (R)emoved, or left (U)nchanged
0.000					
	1	1591.525	3	11.9595	A x1 x2 x3
	* 93	.3052401	3	9.54017	U
	100	.1591525	3	9.566065	U

* alpha and lambda selected by cross-validation.

```
. lassocoef, display(coef, penalized) nolegend
```

	active
x1	1.139476
x2	.4865453
x3	.0958546
_cons	2.659647

Elastic net

- Default is λ 100 point logarithmic grid and $\alpha = 0.5, 0.7, 1.0$
  - here $\alpha = 1.0$ (lasso) so narrow grid to 0.90, 0.95, 1.0
  - optimal $\alpha = 0.95$, $\lambda = 0.2717$, and x1 and x2 selected.

```
. * Elastic net estimation and selected results
. qui elasticnet linear y x1 x2 x3, alpha(0.9(0.05)1) rseed(10101) folds(5)

. lassoknots
```

alpha	ID	lambda	No. of nonzero coef.	CV mean pred. error	Variables (A)dded, (R)emoved, or left (U)nchanged
1.000					
	4	1.450138	1	11.60145	A x1
	13	.6277301	2	9.739804	A x2
	26	.1872925	2	9.434326	U
0.950					
	29	1.591525	1	11.73019	A x1
	38	.688933	2	9.81611	A x2
	* 48	.2717294	2	9.3884	U
	51	.2055533	2	9.425887	U
0.900					
	53	1.675289	1	11.74015	A x1
	62	.7561031	2	9.900317	A x2
	76	.2055533	2	9.431641	U

* alpha and lambda selected by cross-validation.

4.6 Comparison of Shrinkage Estimators

- Compare OLS, Lasso, ridge, elastic net (**in-sample** $n = 40$)

```
. * Estimate various models and store results
. qui regress y x1 x2 x3

. estimates store OLS

. qui lasso linear y x1 x2 x3, selection(cv) folds(5) rseed(10101)

. estimates store LASCV

. qui lasso linear y x1 x2 x3, selection(adaptive) folds(5) rseed(10101)

. estimates store LASADAPT

. qui lasso linear y x1 x2 x3, selection(plugin) folds(5)

. estimates store LASPLUG

. qui elasticnet linear y x1 x2 x3, alpha(0) selection(cv) folds(5) rseed(10101)

. estimates store RIDGECV

. qui elasticnet linear y x1 x2 x3, alpha(0.9(0.05)1) rseed(10101) folds(5)

. estimates store ELASTIC
```

Comparison of Shrinkage Estimators

- Most select x1 and x2, adaptive lasso only x1, ridge all three.
  - lassogof default is penalized coefficients and R^2

```
. * Compare in-sample fit and selected coefficients of various models
. lassogof OLS LASCV LASADAPT LASPLUG RIDGECV ELASTIC

Penalized coefficients
```

Name	MSE	R-squared	Obs
OLS	8.597403	0.2373	40
LASCV	8.679274	0.2300	40
LASADAPT	8.755573	0.2232	40
LASPLUG	10.27195	0.0887	40
RIDGECV	8.70562	0.2277	40
ELASTIC	8.693386	0.2288	40

```
. lassocoef OLS LASCV LASADAPT LASPLUG RIDGECV ELASTIC, display(coef) nolegend
```

	OLS	LASCV	LASADAPT	LASPLUG	RIDGECV	ELASTIC
x1	1.555582	1.206056	1.462431	.3533654	1.011134	1.179972
x2	.4707111	.2715635			.452667	.2705777
x3	-.0256025				.0979251	
_cons	2.531396	0	0	0	0	0

4.7 Shrinkage for logit, probit and poisson

- More generally can apply shrinkage to other objective functions.
- For logit, probit and poisson replace the squared residual by the squared deviance residual
  - deviance residual is used for generalized linear models
- Consider lasso $\widehat{\boldsymbol{\beta}}_\lambda$ of $\boldsymbol{\beta}$ minimizes

$$Q_\lambda(\boldsymbol{\beta}) = \sum_{i=1}^{n} q(y_i, \mathbf{x}_i, \boldsymbol{\beta}) + \lambda \sum_{j=1}^{p} |\beta_j|$$

- Logit: $q(y_i, \mathbf{x}_i, \boldsymbol{\beta}) = \{2[y_i \ln \Lambda(\mathbf{x}_i'\boldsymbol{\beta}) + (1-y_i)\ln(1-\Lambda(\mathbf{x}_i'\boldsymbol{\beta})]\}^2$
- Probit: $q(y_i, \mathbf{x}_i, \boldsymbol{\beta}) = \{2[y_i \ln \Phi(\mathbf{x}_i'\boldsymbol{\beta}) + (1-y_i)\ln(1-\Phi(\mathbf{x}_i'\boldsymbol{\beta})]\}^2$
- Poisson: $q(y_i, \mathbf{x}_i, \boldsymbol{\beta}) = \{2[y_i\mathbf{x}_i'\boldsymbol{\beta} - \exp(\mathbf{x}_i'\boldsymbol{\beta}) - v_i]\}^2$
  - $v_i = 0$ if $y_i = 0$ and $v_i = y_i \ln y_i$ otherwise.

Lasso shrinkage for logit example

- Create dy = y > 3
  - only x1 selected

```
. * Lasso for logit example
. qui generate dy = y > 3

. qui lasso logit dy x1 x2 x3, rseed(10101) folds(5)

. lassoknots
```

ID	lambda	No. of nonzero coef.	CV mean deviance	Variables (A)dded, (R)emoved, or left (U)nchanged
2	.2065674	1	1.407613	A x1
* 24	.0266792	1	1.192646	U
26	.0221495	2	1.192865	A x2
30	.0152668	3	1.194545	A x3
31	.0139106	3	1.195055	U

* lambda selected by cross-validation.

Other user-written Stata commands for LASSO

- `Lassopack` package of Ahrens, Hansen and Schaffer (2020)
  - `cvlasso` for λ chosen by K-fold cross-validation and h-step ahead rolling cross-validation for cross-section, panel and time-series data
  - `rlasso` for theory-driven ('rigorous') penalization for the lasso and square-root lasso for cross-section and panel data
  - `lasso2` for information criteria choice of λ
  - now supplanted by Stata's commands
  - but the Ahrens, Hansen and Schaffer (2020) is a great article to read as it provides a lot of detail.

5. Prediction for Economics

- Microeconometrics focuses on estimation of β or of partial effects (later).
- But in some cases we are directly interested in predicting y
  - for old people predict probability of one-year survival
    - ★ if low then do not have hip replacement surgery.
  - probability of re-offending
    - ★ if low then grant parole to prisoner.
- Mullainathan and Spiess (2017, JEP)
  - consider prediction of housing prices
  - detail how to do this using machine learning methods
  - and then summarize many recent economics ML applications.

5.1 Predict housing prices

- y is log house price in U.S. 2011
  - $n = 51,808$ is sample size
  - $p = 150$ is number of potential regressors.
- Predict using
  - OLS (using all regressors)
  - regression tree
  - LASSO (and not post-LASSO OLS)
  - random forest
  - ensemble: an optimal weighted average of the above methods.
- 1. Train model on 10,000 observations using 8-fold CV.
- 2. Fit preferred model on these 10,000 observations.
- 3. Predict on remaining 41,808 observations
  - and do 500 bootstraps to get 95% CI for R^2.

- Random forest (and subsequent ensemble) does best out of sample
  - training sample is $n = 10,000$ and holdout sample is $n = 41,808$.

Table 1

Performance of Different Algorithms in Predicting House Values

	Prediction performance (R^2)		*Relative improvement over ordinary least squares by quintile of house value*				
Method	*Training sample*	*Hold-out sample*	1st	2nd	3rd	4th	5th
Ordinary least squares	47.3%	41.7% [39.7%, 43.7%]	–	–	–	–	–
Regression tree tuned by depth	39.6%	34.5% [32.6%, 36.5%]	−11.5%	10.8%	6.4%	−14.6%	−31.8%
LASSO	46.0%	43.3% [41.5%, 45.2%]	1.3%	11.9%	13.1%	10.1%	−1.9%
Random forest	85.1%	45.5% [43.6%, 47.5%]	3.5%	23.6%	27.0%	17.8%	−0.5%
Ensemble	80.4%	45.9% [44.0%, 47.9%]	4.5%	16.0%	17.9%	14.2%	7.6%

Details

- Downloadable appendix to the paper gives more details and R code.
- 1. Divide into training $(n = 10,000)$ and hold-out sample $(n = 41,808)$.
- 2. On the training sample do 8-fold cross-validation to get tuning parameter(s) such as λ.
  - If e.g. two tuning parameters then do two-dimensional grid search.
- 3. The prediction function $\widehat{f}(x)$ is estimated using the entire training sample $(n = 10,000)$ with optimal λ.
- 4. Now apply this $\widehat{f}(x)$ to the hold-out sample
  compute MSE$= \frac{1}{41808}\sum_i (y_i - \widehat{f}(x_i))^2$
  hence compute $R^2 = 1 - \frac{\sum_i (y_i - \widehat{f}(x_i))^2}{\sum_i (y_i - \bar{y})^2} = 1 - \frac{MSE}{\frac{1}{41808}(y_i - \bar{y})^2}$.
- 5. A 95% CI for R^2 is obtained by bootstrapping the hold-out sample.

Ensemble Weights

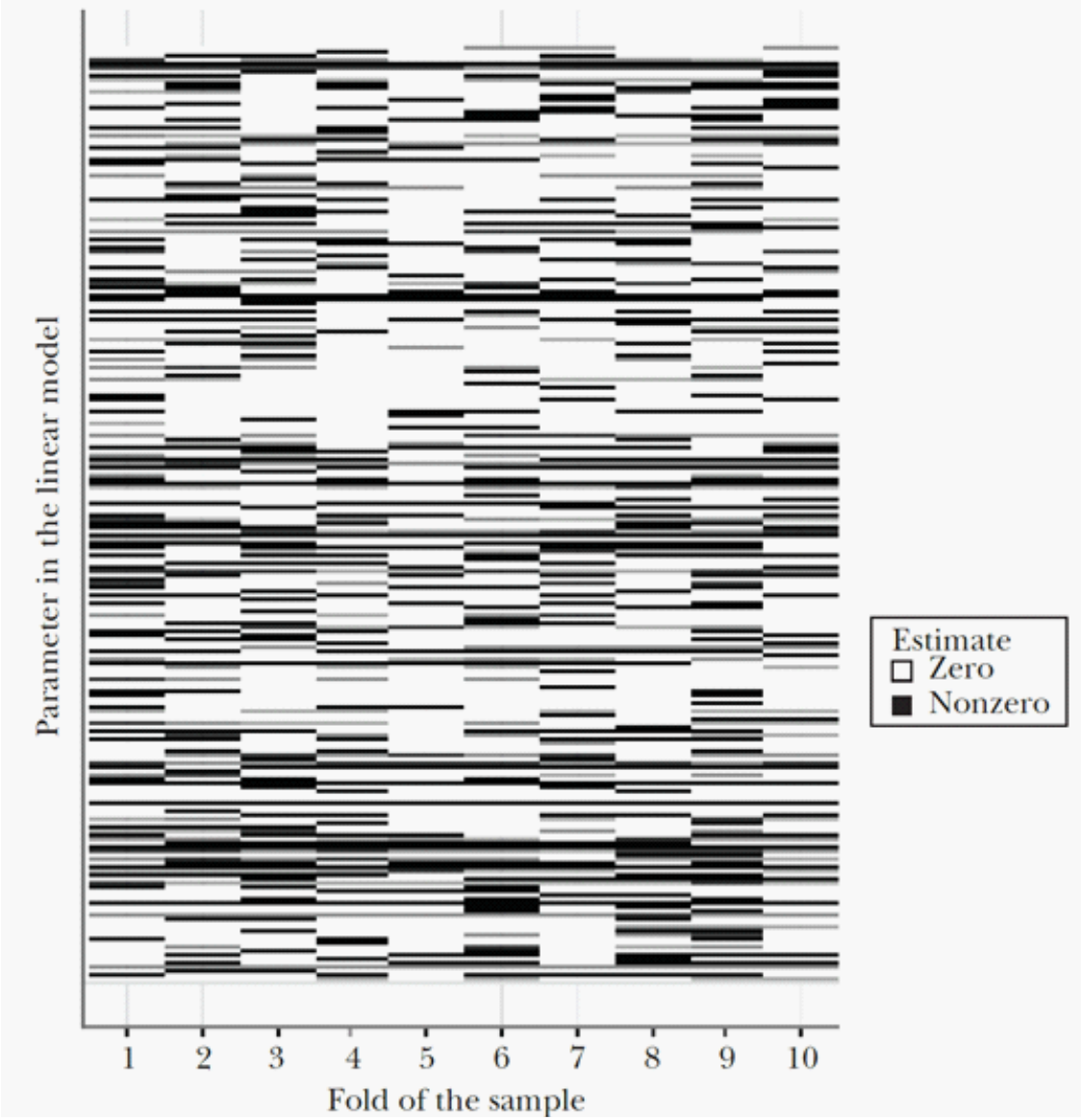
- Ensemble weights are similar to portfolio diversification.
- Example: $X_1 \sim (\mu, \sigma^2)$ independent of $X_2 \sim (\mu, \sigma^2)$
  then
  $Var[(X_1 + X_2)/2] = \frac{1}{4}\{Var[X_1] + Var[X_2]\} = \frac{\sigma^2}{2} < Var[X_1] = \sigma^2$.
  - benefit is less the more correlated are X_1 and X_2.
- So consider a linear combination of predictions.
- For each ML method create 10,000 predictions in the training sample as follows
  - for each of the eight folds estimate (using the optimal tuning parameter(s)) using seven folds and predict on the remaining fold
  - this gives $(10,000 \times 1)$ vectors $\widehat{\mathbf{y}}_{OLS}$, $\widehat{\mathbf{y}}_{REGTREE}$, $\widehat{\mathbf{y}}_{LASSO}$, $\widehat{\mathbf{y}}_{RF}$.
- The ensemble weights are the $\widehat{\alpha}'s$ from the OLS regression in the training sample

$$y_i = \alpha_0 + \alpha_1 \widehat{y}_{OLS,i} + \alpha_2 \widehat{y}_{REGTREE.i} + \alpha_3 \widehat{y}_{LASSO,i} + \alpha_4 \widehat{y}_{RF,i} + u_i.$$

- These ensemble weights are also used in the holdout sample exercise.

Further Details

- LASSO does not pick the "correct" regressors
  - it just gets the correct $\widehat{f}(x)$ especially when regressors are correlated with each other.
- Diagram on next slide shows which of the 150 variables are included in separate models for 10 subsamples
  - there are many variables that appear sometimes but not at other times
    - ★ appearing sometimes in white and sometimes in black.


Parameter in the linear model
1
2
3
4
5
6
7
8
9
10
Fold of the sample
Estimate
Zero
Nonzero

6. Some R Commands

- These are from *An Introduction to Statistical Learning: with Applications in R*. **There may be better commands.**
- Basic regression
  - OLS is lm.fit
  - cross-validation for OLS uses cv.glm()
  - bootstrap uses boot() function in boot library
- Variable selection
  - best subset, forward stepwise and backward stepwise: regsubsets() in leaps library
- Penalized regression
  - ridge regression: glmnet(,alpha=0) function in glmnet library
  - lasso: glmnet(,alpha=1) function in glmnet library
  - CV to get lambda for ridge/lasso: cv.glmnet() in glmnet library

7. Some Python Commands

- Use the scikit-learn package.
- More to come here.

8. References

- ISLR2: Gareth James, Daniela Witten, Trevor Hastie and Robert Tibsharani (2021), *An Introduction to Statistical Learning: with Applications in R*, 2nd Ed., Springer.
- ISLP: Gareth James, Daniela Witten, Trevor Hastie, Robert Tibsharani and Jonathan Taylor (2023), *An Introduction to Statistical Learning: with Applications in Python*, Springer.
  - Free PDF from https://www.statlearning.com/ and $40 softcover book via Springer Mycopy.
- ESL: Trevor Hastie, Robert Tibsharani and Jerome Friedman (2009), *The Elements of Statistical Learning: Data Mining, Inference and Prediction*, Springer.
  - PDF and $40 softcover book at https://link.springer.com/book/10.1007/978-0-387-84858-7.
- Geron2: Aurelien Geron (2022), *Hands-On Machine Learning with Scikit-Learn, Keras and Tensor Flow*, Third edition, O'Reilly
- A. Colin Cameron and Pravin K. Trivedi (2022), *Microeconometrics using Stata*, Second edition, Chapters 28.3-28.4.
- EH: Bradley Efron and Trevor Hastie (2016), *Computer Age Statistical Inference:*

References (continued)

- Sendhil Mullainathan and J. Spiess (2017): "Machine Learning: An Applied Econometric Approach", Journal of Economic Perspectives, Spring, 87-106.
- Hannes Leeb and Benedikt M. Pötscher (2005), "Model selection and Inference", Econometric Theory, 21-59.
- Peter Bickel, Ya'acov Ritov and Alexandre Tsybakov (2009), "Simultaneous Analysis of Lasso and Dantzig Selector", The Annals of Statistics, 1705-1732.
- Alex Belloni and Victor Chernozhukov (2013), "Least Squares after Model Selection in High-Dimensional Sparse Models", Bernoulli, 521-547.
- Achim Ahrens, Christian B. Hansen, Mark E. Schaffer (2020), "lassopack: Model selection and prediction with regularized regression in Stata", The Stata Journal, 20, 176-235 (also ArXiv:1901.05397).
- Domenico Giannone, Michele Lenza, Giorgio E. Primiceri (2021), "Economic Predictions with Big Data: The Illusion of Sparsity," Econometrica, 2409-2437.