Day 1
Ordinary Least Squares and GLS

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Advanced Econometrics
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Based on A. Colin Cameron and Pravin K. Trivedi (2009,2010),
Microeconometrics using Stata (MUS), Stata Press.
and A. Colin Cameron and Pravin K. Trivedi (2005),
Microeconometrics: Methods and Applications (MMA), C.U.P.

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1. Introduction

- OLS for the linear model is the building block for other regression.
- Here we provide
  - model in matrix notation
  - statistical properties
  - hypothesis testing
  - simulations to show consistency and asymptotic normality.
- Additionally
  - More efficient FGLS with heteroskedastic data

Overview

2. Data Example: OLS for doctor visits

- Cross-section data on individuals (from MUS chapter 10).
  - Dependent variable `docvis` is a count. Here do OLS (later Poisson).
  - Begin with data description and summary statistics.

```
. use mus10data.dta, clear

. quietly keep if year02==1

. describe docvis private chronic female income

              storage  display     value
variable name   type   format      label      variable label
-------------------------------------------------------------------------
docvis          int    %8.0g                  number of doctor visits
private         byte   %8.0g                  = 1 if private insurance
chronic         byte   %8.0g                  = 1 if a chronic condition
female          byte   %8.0g                  = 1 if female
income          float  %9.0g                  Income in $ / 1000

. summarize docvis private chronic female income

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
      docvis |      4412    3.957389    7.947601          0        134
     private |      4412    .7853581    .4106202          0          1
     chronic |      4412    .3263826    .4689423          0          1
      female |      4412    .4718948    .4992661          0          1
      income |      4412    34.34018    29.03987    -49.999    280.777
```

- OLS regression with default standard errors: assumes i.i.d error.

```
. * OLS regression with default standard errors
. regress docvis private chronic female income

      Source |       SS       df       MS              Number of obs =    4412
-------------+------------------------------           F(  4,  4407) =  162.29
       Model |  35771.7188     4  8942.92971           Prob > F      =  0.0000
    Residual |   242846.27  4407  55.1046676           R-squared     =  0.1284
-------------+------------------------------           Adj R-squared =  0.1276
       Total |  278617.989  4411  63.1643594           Root MSE      =  7.4233

------------------------------------------------------------------------------
      docvis |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     private |   1.916263   .2881911     6.65   0.000     1.351264    2.481263
     chronic |   4.826799   .2419767    19.95   0.000     4.352404    5.301195
      female |   1.889675   .2286615     8.26   0.000     1.441384    2.337967
      income |    .016018    .004071     3.93   0.000     .0080367    .0239993
       _cons |  -.5647368   .2746696    -2.06   0.040    -1.103227   -.0262465
------------------------------------------------------------------------------
```

- Overall fit poor as $R^2 = 0.13$. Often the case for cross-section data.
- Yet all regressors are stat. significant and have large impact.
  - For income: annual income $\uparrow$ \$10,000 $\Rightarrow$ `income` $\uparrow$ 10 units $\Rightarrow$ `docvis` $\uparrow$ $10 \times 0.016 = 0.16$.

- OLS regression with robust standard errors for OLS estimator
  - preferred at this permits model error to be heteroskedastic

```
. * OLS regression with robust standard errors
. regress docvis private chronic female income, vce(robust)

Linear regression                                      Number of obs =    4412
                                                       F(  4,  4407) =  107.01
                                                       Prob > F      =  0.0000
                                                       R-squared     =  0.1284
                                                       Root MSE      =  7.4233
```

docvis	Coef.	Robust Std. Err.	t	P>|t|	[95% Conf.	Interval]
private	1.916263	.2347443	8.16	0.000	1.456047	2.37648
chronic	4.826799	.3001866	16.08	0.000	4.238283	5.415316
female	1.889675	.2154463	8.77	0.000	1.467292	2.312058
income	.016018	.005606	2.86	0.004	.0050275	.0270085
_cons	-.5647368	.2069188	-2.73	0.006	-.9704017	-.159072

- Same coefficient estimates. Different standard errors.

```
. * Comparison of standard errors
. quietly regress docvis private chronic female income

. estimates store DEFAULT

. quietly regress docvis private chronic female income, vce(robust)

. estimates store ROBUST

. estimates table DEFAULT ROBUST, b(%9.4f) se stats(N r2 F)
```

Variable	DEFAULT	ROBUST
private	1.9163	1.9163
	0.2882	0.2347
chronic	4.8268	4.8268
	0.2420	0.3002
female	1.8897	1.8897
	0.2287	0.2154
income	0.0160	0.0160
	0.0041	0.0056
_cons	-0.5647	-0.5647
	0.2747	0.2069
N	4412.0000	4412.0000
r2	0.1284	0.1284
F	162.2899	107.0104

legend: b/se

- The preferred heteroskedastic-robust standard errors are within 25% of default, sometimes more and sometimes less.

- Hypothesis tests can be implemented using Stata command `test`

$$\begin{aligned} H_0 &: \beta_{\text{private}} = 0, \beta_{\text{chronic}} = 0 \\ H_a &: \text{at least one of } \beta_{\text{private}} \neq 0, \beta_{\text{chronic}} \neq 0. \end{aligned}$$

- Stata post-estimation command `test` yields

```
. * Wald test of restrictions
. quietly regress docvis private chronic female income, vce(robust) noheader

. test (private = 0) (chronic = 0)

 ( 1)  private = 0
 ( 2)  chronic = 0

       F(  2,  4407) =  165.11
            Prob > F =    0.0000
```

- Reject H_0 at level 0.05 since $p < 0.05$ or $165.11 > F_{.05}(2, 4407) = 3.00$ using `invFtail(2,4407,.05)`.

3. OLS: Definition in matrix notation

- For the i^{th} observation

$$y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \cdots + \beta_K x_{Ki} + u_i$$

  - Usually $x_{1i} = 1$ (an intercept).

- Introduce vector and matrix representation.
  - Regressor vector $\mathbf{x}_i$ and parameter vector $\boldsymbol{\beta}$ are $K \times 1$ column vectors.

$$\underset{(K\times 1)}{\mathbf{x}_i} = \begin{bmatrix} x_{1i} \\ \vdots \\ x_{Ki} \end{bmatrix} \quad \text{and} \quad \underset{(K\times 1)}{\boldsymbol{\beta}} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_K \end{bmatrix}.$$

$$\mathbf{x}_i'\boldsymbol{\beta} = \begin{bmatrix} x_{1i} & \cdots & x_{Ki} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_K \end{bmatrix} = \beta_1 x_{1i} + \beta_2 x_{2i} + \cdots + \beta_K x_{Ki}$$

  - Note that all vectors are defined to be column vectors

- For the i^{th} observation

$$y_i = \mathbf{x}_i'\boldsymbol{\beta} + u_i.$$

- Now combine all N observations from sample $\{(y_i, \mathbf{x}_i), \, i = 1, ..., N.\}$
- The linear regression model is

$$\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1'\beta \\ \vdots \\ \mathbf{x}_N'\beta \end{bmatrix} + \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}$$

- This is

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{u}$$

where

$$\underset{(N\times 1)}{\mathbf{y}} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} \qquad \underset{(N\times K)}{\mathbf{X}} = \begin{bmatrix} \mathbf{x}_1' \\ \vdots \\ \mathbf{x}_N' \end{bmatrix} \qquad \underset{(N\times 1)}{\mathbf{u}} = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}.$$

- The OLS estimator derived below is

$$\widehat{\beta}_{\text{OLS}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}.$$

OLS: matrix notation example

- Example: $N = 4$ with (x, y) equal to $(1, 1)$, $(2, 3)$, $(2, 4)$, and $(3, 4)$.
- Then $\mathbf{y}$ is 4×1 and $\mathbf{X}$ is 4×2 with

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 4 \end{bmatrix}; \quad \mathbf{X} = \begin{bmatrix} \mathbf{x}_1' \\ \mathbf{x}_2' \\ \mathbf{x}_3' \\ \mathbf{x}_4' \end{bmatrix} = \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \\ x_{13} & x_{23} \\ x_{14} & x_{24} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

- So (see appendix for detailed computation)

$$\widehat{\beta}_{\text{OLS}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \begin{bmatrix} 4 & 8 \\ 8 & 18 \end{bmatrix}^{-1} \begin{bmatrix} 12 \\ 27 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$$

- Intercept $\widehat{\beta}_1 = 0$ and slope coefficient $\widehat{\beta}_2 = 1.5$.

Derivation of formula for OLS estimator

- The OLS estimator minimizes the sum of squared errors

$$Q(\boldsymbol{\beta}) = \sum_{i=1}^{N} u_i^2 = \sum_{i=1}^{N} (y_i - \mathbf{x}_i'\boldsymbol{\beta})^2.$$

- The first-order conditions (f.o.c.) are

$$\frac{\partial Q(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = -2\sum_{i=1}^{N} \mathbf{x}_i (y_i - \mathbf{x}_i'\boldsymbol{\beta}) = -2\mathbf{X}'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) = \mathbf{0}.$$

- Then

$$\begin{array}{rcl} & \mathbf{X}'(\mathbf{y} - \mathbf{X}\beta) = \mathbf{0} & \text{from f.o.c.} \\ \Rightarrow & \mathbf{X}'\mathbf{y} = \mathbf{X}'\mathbf{X}\beta & K \text{ linear equations in } K \text{ unknowns } \boldsymbol{\beta} \\ \Rightarrow & \beta = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} & \text{if the inverse exists (i.e. rank}[X] = K) \end{array}$$

- So

$$\widehat{\beta}_{\text{OLS}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \left(\sum_{i=1}^{N} \mathbf{x}_i\mathbf{x}_i'\right)^{-1} \sum_{i=1}^{N} \mathbf{x}_i y_i.$$

4. OLS Properties: Summary

- $\widehat{\boldsymbol{\beta}}_{\text{OLS}}$ is always estimable, provided $\text{rank}[X] = K$.
- But properties of $\widehat{\boldsymbol{\beta}}_{\text{OLS}}$ depend on the true model
  - called the data generating process (d.g.p.)
- Essential result:
  - If the d.g.p. is correctly specified and the error u_i is uncorrelated with regressors $\mathbf{x}_i$
  - Then
    (1) $\widehat{\boldsymbol{\beta}}$ is consistent for $\boldsymbol{\beta}$
    (2) $\widehat{\boldsymbol{\beta}}$ is normally distributed in large samples ("asymptotically")
    (3) Variance of $\widehat{\boldsymbol{\beta}}$ varies with assumptions on error u_i
      - ★ default: u_i are independent $(0, \sigma^2)$
      - ★ heteroskedastic: u_i are independent $(0, \sigma_i^2)$
      - ★ clustered: u_i are correlated within cluster, uncorrelated across cluster
      - ★ HAC: u_i are serially correlated (u_i are correlated with u_{i-1})

OLS Properties

- If the d.g.p. is $\mathbf{y} = \mathbf{X}\beta + \mathbf{u}$ then

$$\begin{aligned}
\widehat{\beta}_{\text{OLS}} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} \\
&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{X}\beta + \mathbf{u}) \\
&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u} \\
&= \beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u} \\
&= \beta + \left(\sum_i \mathbf{x}_i\mathbf{x}_i'\right)^{-1} \sum_i \mathbf{x}_i u_i
\end{aligned}$$

- So assumptions on $\mathbf{x}_i$ and u_i are crucial.

OLS Finite Sample Properties

- If $\mathbf{u} \sim \mathcal{N}[\mathbf{0}, \Omega]$ and regressors $\mathbf{X}$ are fixed (nonstochastic) then

$$\begin{aligned}\widehat{\beta} &= \beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u} \\ &\sim \beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' \times \mathcal{N}[\mathbf{0}, \Omega] \\ &\sim \mathcal{N}[\beta,\ (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Omega\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}]\end{aligned}$$

  - using linear transformation of the normal is normal $\mathbf{z} \sim \mathcal{N}[\mu, \Omega] \Longrightarrow \mathbf{Az} + \mathbf{b} \sim \mathcal{N}[\mathbf{A}\mu + \mathbf{b}, \mathbf{A}\Omega\mathbf{A}']$.
- We instead use asumptotic theory
  - this permits $\mathbf{u}$ to be nonnormal distributed.
  - but does require a large sample so $N \rightarrow \infty$.

OLS Consistency

- Consistency
  - Means that the probability limit (plim) of $\widehat{\boldsymbol{\beta}}$ equals $\boldsymbol{\beta}$
  - That is: $\lim_{N \to \infty} \Pr[|\,\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}| < \varepsilon] = 1$ for any $\varepsilon > 0$.
- We have (using results below)

$$
\begin{aligned}
\text{plim}\,\widehat{\boldsymbol{\beta}} &= \text{plim}\{\boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}\} \\
&= \text{plim}\,\boldsymbol{\beta} + \text{plim}\left\{\left(\sum_i \mathbf{x}_i\mathbf{x}_i'\right)^{-1} \sum_i \mathbf{x}_i u_i\right\} \\
&= \text{plim}\,\boldsymbol{\beta} + \text{plim}\left(\tfrac{1}{N}\sum_i \mathbf{x}_i\mathbf{x}_i'\right)^{-1} \times \text{plim}\,\tfrac{1}{N}\sum_i \mathbf{x}_i u_i \\
&= \boldsymbol{\beta} + \left(\text{plim}\,\tfrac{1}{N}\sum_i \mathbf{x}_i\mathbf{x}_i'\right)^{-1} \times \mathbf{0} \\
&= \boldsymbol{\beta}
\end{aligned}
$$

  - $\text{plim}\{\mathbf{A}_N \times \mathbf{b}_N\} = \text{plim}\,\mathbf{A}_N \times \text{plim}\,\mathbf{b}_N$ if the plim's are constants
  - The plim's exist using laws of large numbers (as averages)
  - For $\text{plim}\,\frac{1}{N}\sum_i \mathbf{x}_i u_i = \mathbf{0}$ the key assumption is $\mathrm{E}[u_i|\mathbf{x}_i] = 0$.

OLS Limit Distribution

- $\widehat{\boldsymbol{\beta}}$ has limit distribution with all mass at $\boldsymbol{\beta}$ (since $\widehat{\boldsymbol{\beta}} \xrightarrow{p} \boldsymbol{\beta}$).
  - To get a nondegenerate distribution inflate $\widehat{\boldsymbol{\beta}}$ by $\sqrt{N}$.
- Then limit normal distribution is

$$\begin{aligned}
\sqrt{N}(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}) &= \left(\tfrac{1}{N}\textstyle\sum_i \mathbf{x}_i\mathbf{x}_i'\right)^{-1} \tfrac{1}{\sqrt{N}}\textstyle\sum_i \mathbf{x}_i u_i \\
&\xrightarrow{d} \operatorname{plim}\left(\tfrac{1}{N}\textstyle\sum_i \mathbf{x}_i\mathbf{x}_i'\right)^{-1} \times \mathcal{N}[\mathbf{0},\mathbf{B}] \text{ for some } \mathbf{B} \\
&\xrightarrow{d} \mathcal{N}\left[\mathbf{0}, \operatorname{plim}\left(\tfrac{1}{N}\textstyle\sum_i \mathbf{x}_i\mathbf{x}_i'\right)^{-1} \times \mathbf{B} \times \operatorname{plim}\left(\tfrac{1}{N}\textstyle\sum_i \mathbf{x}_i\mathbf{x}_i'\right)^{-1}\right]
\end{aligned}$$

  - If $\mathbf{H}_N \xrightarrow{p} \mathbf{H}$ and $\mathbf{b}_N \xrightarrow{d} \mathcal{N}[\boldsymbol{\mu},\Omega]$ then $\mathbf{H}_N\mathbf{b}_N \xrightarrow{p} \mathcal{N}[\mathbf{H}\boldsymbol{\mu},\ \mathbf{H}\Omega\mathbf{H}']$
  - $\frac{1}{\sqrt{N}}\sum_i \mathbf{x}_i u_i \xrightarrow{d} \mathcal{N}[\mathbf{0},\mathbf{B}]$ by a central limit theorem
  - $\mathbf{B} = \operatorname{plim}\left(\frac{1}{\sqrt{N}}\sum_i \mathbf{x}_i u_i\right)\left(\frac{1}{\sqrt{N}}\sum_i \mathbf{x}_i u_i\right)' = \operatorname{plim}\frac{1}{N}\sum_i\sum_j u_i u_j \mathbf{x}_i\mathbf{x}_j'$

OLS Asymptotic Distribution

- All we need for theory is the previous result.
  - but rescale from $\sqrt{N}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ to $\widehat{\boldsymbol{\beta}}$ for "friendlier" looking results
  - drop plims and replace $\mathbf{B}$ by a consistent estimate $\widehat{\mathbf{B}}$
- The so-called "asymptotic distribution" is

$$\widehat{\boldsymbol{\beta}} \overset{a}{\sim} \mathcal{N}\left[\boldsymbol{\beta}, \left(\sum_{i=1}^{N} \mathbf{x}_i \mathbf{x}_i'\right)^{-1} \times N\widehat{\mathbf{B}} \times \left(\sum_{i=1}^{N} \mathbf{x}_i \mathbf{x}_i'\right)^{-1}\right]$$

  - Usually $\mathbf{B} = \mathrm{Var}[\frac{1}{\sqrt{N}}\mathbf{X}'\mathbf{u}] = \mathrm{Var}[\frac{1}{\sqrt{N}}\sum_i \mathbf{x}_i u_i]$
  - For independent heteroskedastic errors $\widehat{\mathbf{B}} = \frac{1}{N}\sum_i \widehat{u}_i^2 \mathbf{x}_i \mathbf{x}_i'$.

White Estimate of VCE

- Most often used: requires data to be independent over i.
- Then $\mathbf{B} = \text{plim}\, \frac{1}{N} \sum_i \sum_j u_i u_j \mathbf{x}_i \mathbf{x}_j' = \text{plim}\, \frac{1}{N} \sum_i u_i^2 \mathbf{x}_i \mathbf{x}_i'$.
- White (1980) showed that can use $\widehat{\mathbf{B}} = \frac{1}{N} \sum_i \widehat{u}_i^2 \mathbf{x}_i \mathbf{x}_i'$.
- Yields the heteroskedastic-consistent estimate of the variance-covariance matrix of the OLS estimator (VCE)

$$\widehat{\mathrm{V}}_{\text{robust}}[\widehat{\boldsymbol{\beta}}] = \left(\sum_{i=1}^N \mathbf{x}_i \mathbf{x}_i' \right)^{-1} \sum_{i=1}^N \widehat{u}_i^2 \mathbf{x}_i \mathbf{x}_i' \left(\sum_{i=1}^N \mathbf{x}_i \mathbf{x}_i' \right)^{-1}$$

  - $\widehat{u}_i = y_i - \mathbf{x}_i' \widehat{\boldsymbol{\beta}}$
  - Leads to "heteroskedastic robust" or "robust" standard errors.
  - In Stata this is option `vce(robust)` for cross-section commands

Other Estimates of VCE

- **Default**: Independent homoskedastic errors: $\mathrm{V}[u_i|\mathbf{x}_i] = \sigma^2$

$$\widehat{\mathrm{V}}[\widehat{\boldsymbol{\beta}}] = s^2 \left(\sum_{i=1}^{N} \mathbf{x}_i \mathbf{x}_i' \right)^{-1}; \; s^2 = \frac{1}{N-K} \sum_i \widehat{u}_i^2$$

  - Simplification as then $\mathbf{B} = \text{plim}\, \frac{1}{N} \sum_i u_i^2 \mathbf{x}_i \mathbf{x}_i' = \sigma^2 \,\text{plim} \sum_i \mathbf{x}_i \mathbf{x}_i'$

- **Cluster robust**: Errors correlated within cluster but independent across cluster.

$$\widehat{\mathrm{V}}[\widehat{\boldsymbol{\beta}}] = \left(\sum_{g=1}^{G} \mathbf{X}_g \mathbf{X}_g{}' \right)^{-1} \sum_{g=1}^{G} \mathbf{X}_g \widehat{\mathbf{u}}_g \widehat{\mathbf{u}}_g{}' \mathbf{X}_g \left(\sum_{g=1}^{G} \mathbf{X}_g \mathbf{X}_g{}' \right)^{-1}.$$

  - Here observations are stacked in cluster g as $\mathbf{y}_g = \mathbf{X}_g \boldsymbol{\beta} + \mathbf{u}_g$.
  - In Stata this is option `vce(cluster id)` for cross-section commands
  - and is option `vce(robust)` for most `xt` panel commands.

- **Heteroskedasticity and autocorrelation (HAC) robust**: time series
  - Not covered here but extends White to an MA(q) error.

5. Generalized least squares (GLS) Overview

- OLS is efficient (best linear unbiased estimator) if errors are i.i.d. so that $\mathsf{V}[\mathbf{u}|\mathbf{X}] = \sigma^2\mathbf{I}$.
  - In practice errors are rarely i.i.d.
- So we usually do OLS and obtain robust VCE that permits $\mathsf{V}[\mathbf{u}|\mathbf{X}] \neq \sigma^2\mathbf{I}$
  - could be heteroskedastic robust, cluster-robust, HAC, ...
- More efficient feasible GLS (FGLS) assumes a model for $\mathsf{V}[\mathbf{u}|\mathbf{X}]$
  - yields more precise estimates (smaller standard errors and bigger t-statistics)
  - but then obtain robust VCE that allows for misspecified model for $\mathsf{V}[\mathbf{u}|\mathbf{X}]$.
  - called weighted LS or working matrix LS.

Generalized least squares (GLS)

- Suppose $\mathrm{V}[\mathbf{u}|\mathbf{X}] = \Omega$ where Ω is known
  - and $\mathbf{y} = \mathbf{X}\beta + \mathbf{u}$, $\mathrm{E}[\mathbf{u}|\mathbf{X}] = \mathbf{0}$ as before.
- The generalized least squares estimator is efficient:

$$\widehat{\beta}_{\mathrm{GLS}} = (\mathbf{X}'\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\Omega^{-1}\mathbf{y}.$$

- Derivation:
  - Premultiply $\mathbf{y} = \mathbf{X}\beta + \mathbf{u}$ by $\Omega^{-1/2}$ so

$$\Omega^{-1/2}\mathbf{y} = \Omega^{-1/2}\mathbf{X}\beta + \Omega^{-1/2}\mathbf{u}.$$

  - This model has i.i.d. errors since $\mathrm{V}[\Omega^{-1/2}\mathbf{u}|\mathbf{X}] = \mathrm{E}[(\Omega^{-1/2}\mathbf{u})(\Omega^{-1/2}\mathbf{u})'|\mathbf{X}] = \Omega^{-1/2}\Omega\Omega^{-1/2} = \mathbf{I}_N$.
  - Then GLS is OLS in this transformed model:

$$\begin{aligned} \widehat{\beta}_{\mathrm{GLS}} &= [(\Omega^{-1/2}\mathbf{X})'(\Omega^{-1/2}\mathbf{X})](\Omega^{-1/2}\mathbf{X})'(\Omega^{-1/2}\mathbf{y}) \\ &= (\mathbf{X}'\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\Omega^{-1}\mathbf{y}. \end{aligned}$$

Feasible generalized least squares (FGLS)

- To implement GLS we need a consistent estimate of Ω.
  Assume a model for $\Omega = \Omega(\gamma)$, estimate $\widehat{\gamma} \xrightarrow{p} \gamma$,
  and form $\widehat{\Omega} = \Omega(\widehat{\gamma}) \xrightarrow{p} \Omega$.
- The feasible GLS estimator (FGLS) is

$$\widehat{\boldsymbol{\beta}}_{\text{FGLS}} = (\mathbf{X}'\widehat{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}'\widehat{\Omega}^{-1}\mathbf{y},$$

  and then

$$\widehat{\boldsymbol{\beta}}_{\text{FGLS}} \overset{a}{\sim} \mathcal{N}\left[\boldsymbol{\beta}, (\mathbf{X}'\widehat{\Omega}^{-1}\mathbf{X})^{-1}\right].$$

- Examples:
  - Heteroskedasticity: $\mathrm{V}[u_i|\mathbf{x}_i] = \exp(\mathbf{z}_i'\gamma)$
  - Seemingly unrelated equations: $y_{ig} = \mathbf{x}_{ig}'\boldsymbol{\beta}_g + u_{ig}$, $g = 1, ..., G$.
    u_{ig} independent over i and homoskedastic with $\mathrm{Cov}[u_{ig}, u_{ih}] = \sigma_{gh}$.
  - Systems of equations: SUR with $\boldsymbol{\beta}_g = \boldsymbol{\beta}$.
  - Panel data: random effects estimator.

Weighted least squares (WLS)

- Now do FGLS but allow for possibility that model for $\mathrm{V}[\mathbf{u}|\mathbf{X}]$ is incorrectly specified
  - So then obtain robust VCE for FGLS.
- Distinguish between
  - the assumed (working) error variance matrix, denoted $\Sigma = \Sigma(\gamma)$ with estimate $\widehat{\Sigma} = \Sigma(\widehat{\gamma})$.
  - the true (unknown) error variance matrix Ω
- The weighted least squares (WLS) estimator is

$$\widehat{\boldsymbol{\beta}}_{\mathrm{WLS}} = (\mathbf{X}'\widehat{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\widehat{\Sigma}^{-1}\mathbf{y}.$$

- Asymptotically $\widehat{\boldsymbol{\beta}}_{\mathrm{WLS}} \overset{a}{\sim} \mathcal{N}[\boldsymbol{\beta}, \mathrm{V}[\widehat{\boldsymbol{\beta}}]]$ where robust VCE is

$$\widehat{\mathrm{V}}[\widehat{\boldsymbol{\beta}}] = (\mathbf{X}'\widehat{\Sigma}^{-1}\mathbf{X})^{-1}(\mathbf{X}'\widehat{\Sigma}^{-1}\widehat{\Omega}\widehat{\Sigma}^{-1}\mathbf{X})^{-1}(\mathbf{X}'\widehat{\Sigma}^{-1}\mathbf{X})^{-1},$$

  - for cross-section data $\widehat{\Omega} = \mathrm{Diag}[(y_i - \mathbf{x}_i'\widehat{\boldsymbol{\beta}}_{\mathrm{WLS}})^2]$.

Hypothesis test of single restriction

- Consider test of a single restriction, for notational simplicity β

$$\begin{aligned} H_0 &: \beta = \beta^* \\ H_a &: \beta \neq \beta^*. \end{aligned}$$

- A Wald test rejects H_0 if $\widehat{\beta}$ differs greatly from β^*.
- Define $\sigma_{\widehat{\beta}}$ to be the asymptotic standard deviation of $\widehat{\beta}$. Then

$$\begin{array}{rrcl} & \widehat{\beta} & \overset{a}{\sim} \mathcal{N}[\beta, \sigma^2_{\widehat{\beta}}] & \text{for unknown } \beta \\ \Rightarrow & \frac{\widehat{\beta}-\beta}{\sigma_{\widehat{\beta}}} & \overset{a}{\sim} \mathcal{N}[0, 1] & \text{standardizing} \\ \Rightarrow & z_j = \frac{\widehat{\beta}-\beta^*}{\sigma_{\widehat{\beta}}} & \overset{a}{\sim} \mathcal{N}[0, 1] & \text{under } H_0 : \beta = \beta^* \end{array}$$

- To implement this, replace $\sigma_{\widehat{\beta}}$ by $s_{\widehat{\beta}}$, the standard error of $\widehat{\beta}$.
  - This makes no difference asymptotically (so still $\mathcal{N}[0, 1]$).

- The Wald z-statistic is

$$z_j = \frac{\widehat{\beta} - \beta^*}{s_{\widehat{\beta}}} \overset{a}{\sim} \mathcal{N}[0,\ 1] \quad \text{under } H_0 : \beta = \beta^*$$

- Implementation by two equivalent methods
  - Test using p-values: reject H_0 at level 0.05 if

$$p = \Pr[|Z| > |z_j|] < 0.05, \quad \text{where } Z \sim \mathcal{N}[0, 1].$$

  - Test using critical values: reject H_0 at level 0.05 if

$$|z_j| > z_{.025} = 1.96.$$

- Many packages such as Stata use $T(N-k)$ rather than $\mathcal{N}[0,1]$
  - More conservative (less likely to reject H_0)
  - Exact in unlikely special case that $u_i \sim \mathcal{N}[0, \sigma^2]$.

Confidence interval

- A $100(1-\alpha)\%$ confidence interval for β is

$$\widehat{\beta} \pm z_{\alpha/2} \times s_{\widehat{\beta}}.$$

  - in particular a 95% confidence interval is $\widehat{\beta} \pm 1.96 s_{\widehat{\beta}}$.
  - can replace $z_{\alpha/2}$ by $T_{N-k;\alpha/2}$ for better finite sample performance

Hypothesis test of multiple linear restrictions

- Now consider test of several restrictions
  - e.g. Test $H_0 : \beta_2 = 0$, $\beta_3 = 0$ against H_a: at least one $\neq 0$.
- In matrix algebra we test

$$\begin{array}{rl} & H_0 : \mathbf{R}\boldsymbol{\beta} = \mathbf{r} \\ \text{against} & H_a : \mathbf{R}\boldsymbol{\beta} \neq \mathbf{r}. \end{array}$$

- Example: Test $H_0 : \beta_2 = 0$, $\beta_3 = 0$ against H_a: at least one $\neq 0$

$$\begin{bmatrix} \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $\underset{(2\times K)}{\mathbf{R}} \times \underset{(K\times 1)}{\boldsymbol{\beta}} = \underset{(2\times 1)}{\mathbf{r}}$

- A Wald test rejects $H_0 : \mathbf{R}\boldsymbol{\beta} = \mathbf{r}$ if $\mathbf{R}\widehat{\boldsymbol{\beta}} - \mathbf{r}$ differs greatly from $\mathbf{0}$.
- Now $\mathbf{R}\widehat{\boldsymbol{\beta}} - \mathbf{r}$ is normal as linear combination of normals is normal.

$$
\begin{array}{rrl}
 & \widehat{\boldsymbol{\beta}} & \overset{a}{\sim} \mathcal{N}[\boldsymbol{\beta}, \mathrm{V}[\widehat{\boldsymbol{\beta}}]] \\
\Rightarrow & \mathbf{R}\widehat{\boldsymbol{\beta}} - \mathbf{r} & \overset{a}{\sim} \mathcal{N}[\mathbf{R}\boldsymbol{\beta} - \mathbf{r}, \mathbf{R}\mathrm{V}[\widehat{\boldsymbol{\beta}}]\mathbf{R}'] \\
\Rightarrow & \mathbf{R}\widehat{\boldsymbol{\beta}} - \mathbf{r} & \overset{a}{\sim} \mathcal{N}[\mathbf{0}, \mathbf{R}\mathrm{V}[\widehat{\boldsymbol{\beta}}]\mathbf{R}'] \quad \text{under } H_0 \\
\Rightarrow & (\mathbf{R}\widehat{\boldsymbol{\beta}} - \mathbf{r})'[\mathbf{R}\mathrm{V}[\widehat{\boldsymbol{\beta}}]\mathbf{R}']^{-1}(\mathbf{R}\widehat{\boldsymbol{\beta}} - \mathbf{r}) & \sim \chi^2(h) \text{ under } H_0
\end{array}
$$

  - The last step converts to chi-square using the result

$$
\mathbf{z} \sim \mathcal{N}[\mathbf{0}, \Omega] \quad \Rightarrow \quad \mathbf{z}'\Omega^{-1}\mathbf{z} \sim \chi^2(\dim[\Omega]).
$$

- To implement this test, replace $\mathrm{V}[\widehat{\boldsymbol{\beta}}]$ by $\widehat{\mathrm{V}}[\widehat{\boldsymbol{\beta}}]$.
  - This makes no difference asymptotically.

- The Wald chi-squared statistic is

$$\mathrm{W} = (\mathbf{R}\widehat{\beta} - \mathbf{r})'[\mathbf{R}\widehat{\mathrm{V}}[\widehat{\beta}]\mathbf{R}']^{-1}(\mathbf{R}\widehat{\beta} - \mathbf{r}) \overset{a}{\sim} \chi^2(h) \text{ under } H_0$$

- Implementation by two equivalent methods
  - Test using p-values: reject H_0 at level 0.05 if

$$p = \Pr[\chi^2(h) > \mathrm{W}] < 0.05.$$

  - Test using critical-values: reject H_0 at level 0.05 if

$$\mathrm{W} > \chi^2_{.05}(h).$$

- The alternative Wald F-test statistic is

$$\mathrm{F} = \frac{\mathrm{W}}{h} \sim F(h, N-k) \text{ under } H_0$$

  - Makes no difference asymptotically as $F(h, N) \to \chi^2(h)/h$ as $N \to \infty$.
  - More conservative (less likely to reject H_0)
  - Exact in unlikely special case that $u_i \sim \mathcal{N}[0, \sigma^2]$.

Further test details

- Wald test is the commonly-used method to test H_0 against H_a.
  - Estimate β without imposing H_0.
  - Then ask does $\widehat{\beta}$ approximately satisfy H_0?
- The other two test methods used at times are
  - Likelihood ratio test: Estimate under both H_0 & H_a and compare $\ln L$.
  - Lagrange multiplier or score test: Estimate under H_a only.
  - Asymptotically equivalent to Wald under H_0 and local alternatives
  - Choice is mainly one of convenience, though Wald does have the weakness of lack of invariance to reparameterization.
- Also as already noted for Wald test
  - asymptotic theory: use Z and $\chi^2(q)$
  - better finite sample approximation: use $T(N-k)$ and $F(q, N-k)$
  - even better still: bootstrap with asymptotic refinement.

7. Simulations: OLS consistency and asymptotic normality

- D.g.p.: $y_i = \beta_1 + \beta_2 x_i + u_i$ where $x_i \sim \chi^2(1)$ and $\beta_1 = 1$, $\beta_2 = 2$. Error: $u_i \sim \chi^2(1) - 1$ is skewed with mean 0 and variance 2.

```
. * Small sample: parameters differ from dgp values
. clear all

. quietly set obs 30

. set seed 10101

. quietly generate double x = rchi2(1)

. quietly generate y = 1 + 2*x + rchi2(1)-1     // demeaned chi^2 error

. regress y x, noheader
```

y	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
x	2.713073	.5743189	4.72	0.000	1.536634	3.889512
_cons	1.150439	.6148461	1.87	0.072	-.1090161	2.409894

- For $N = 30$: $\widehat{\beta}_2 = 2.713$ differs appreciably from $\beta_2 = 2.000$.
  - This is due to sampling error as $\text{se}[\widehat{\beta}_2] = 0.574$.

- How to verify consistency: set N very large.

```
. * Consistency: Large sample: parameters are very close to dgp values
. clear all

. quietly set obs 100000

. set seed 10101

. quietly generate double x = rchi2(1)

. quietly generate y = 1 + 2*x + rchi2(1)-1     // demeaned chi^2 error

. regress y x, noheader
```

y	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
x	1.998675	.0031725	630.00	0.000	1.992457	2.004893
_cons	1.005819	.0054945	183.06	0.000	.9950495	1.016588

- For $N = 100,000$: $\widehat{\beta}_2 = 1.999$ is very close to $\beta_2 = 2.000$.

- How to check asymptotic results: compute $\widehat{\beta}$ many times.

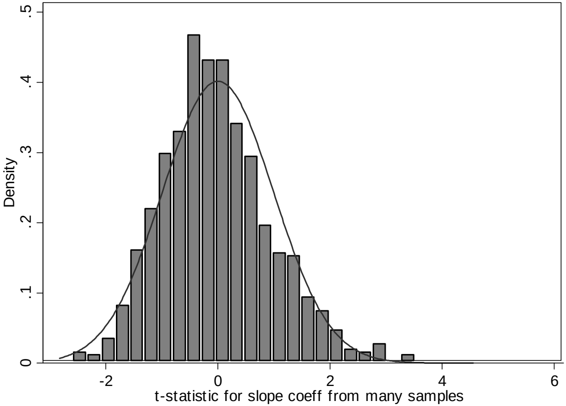
```
. * Central limit theorem
. * Write program to obtain betas for one sample of size numobs (= 150)
. program chi2data, rclass
  1.     version 10.1
  2.     drop _all
  3.     set obs $numobs
  4.     generate double x = rchi2(1)
  5.     generate y = 1 + 2*x + rchi2(1)-1        // demeaned chi^2 error
  6.     regress y x
  7.     return scalar b2 =_b[x]
  8.     return scalar se2 = _se[x]
  9.     return scalar t2 = (_b[x]-2)/_se[x]
 10.     return scalar r2 = abs(return(t2))>invttail($numobs-2,.025)
 11.     return scalar p2 = 2*ttail($numobs-2,abs(return(t2)))
 12. end

. * Run this program 1,000 times to get 1,000 betas etcetera
. * Results differ from MUS (2008) as MUS did not reset the seed to 10101
. * First define global macro numobs for sample size
. global numobs 150

. set seed 10101

. quietly simulate b2f=r(b2) se2f=r(se2) t2f=r(t2) reject2f=r(r2) p2f=r(p2),  ///
>    reps(1000) saving(chi2datares, replace) nolegend nodots: chi2data
```

- Then look at the distribution of these $\widehat{\beta}' s$ and test statistics.

```
. * Summarize the 1,000 sample means
. summarize b2f se2f t2 reject2f p2f

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
         b2f |      1000    2.000506      .08427   1.719513    2.40565
        se2f |      1000    .0839776    .0172588   .0415919    .145264
         t2f |      1000    .0028714    .9932668  -2.824061   4.556576
    reject2f |      1000        .046    .2095899          0          1
         p2f |      1000    .5175818    .2890325   .0000108   .9997772

. mean b2f se2f t2 reject2f p2f

Mean estimation                     Number of obs    =    1000

--------------------------------------------------------------
             |       Mean   Std. Err.     [95% Conf. Interval]
-------------+------------------------------------------------
         b2f |   2.000506   .0026649      1.995277    2.005735
        se2f |   .0839776   .0005458      .0829066    .0850486
         t2f |   .0028714   .0314099     -.0587655    .0645082
    reject2f |       .046   .0066278       .032994     .059006
         p2f |   .5175818     .00914       .499646    .5355177
--------------------------------------------------------------
```

- For $S = 1,000$ simulations each with sample size $N = 150$.
  - $\widehat{\beta}_2^{(1)}, \widehat{\beta}_2^{(2)}, \ldots, \widehat{\beta}_2^{(1000)}$ has distn. with mean 2.001 close to $\beta_2 = 2.000$
  - and standard deviation 0.089 close to $\sqrt{1/150} = 0.082$
    - ★ using $\mathrm{V}[\widehat{\beta}_2] \simeq (\sigma_u^2/\mathrm{V}[x_i])/N = (2/2)/150 = 1/150$.

- Test $\beta_2 = 2$ using $z = (\widehat{\beta}_2 - \beta_2)/\text{se}[\widehat{\beta}_2] = (\widehat{\beta}_2 - 2.0)/\text{se}[\widehat{\beta}_2]$ to test $H_0 : \beta_2 = 2$.
  Histogram and kernel density estimate for $z_1, z_2, \ldots, z_{1000}$.



- Not quite standard normal: $N = 150$ is still not large enough for CLT.

- How to verify that standard errors are correctly estimated.
  - The average of the computed standard errors of $\widehat{\beta}_2$ is 0.0839 (see `mean` of `se2f`)
  - This is close to the simulation estimate of se$[\widehat{\beta}_2]$ of 0.0842 (see `Std.Dev.` of `b2f`)
  - Aside: Actually for this dgp expect $\sqrt{1/150} \simeq 0.082$ using $\mathrm{V}[\widehat{\beta}_2] \simeq (\sigma_u^2/\mathrm{V}[x_i])/N = (2/2)/150 = 1/150$)
- How to verify that test has correct size.
  - The Wald test of $H_0 : \beta_2 = 2$ at level 0.05 has actual size 0.046 (see `mean` of `reject2f`)
  - This is close enough as a 95% simulation interval when $S = 1000$ is

$$0.05 \pm 1.96 \times \sqrt{0.05 \times 0.95/1000} = 0.05 \pm 1.96 \times 0.007 = (0.046, 0.064).$$

8. Stata commands

- Command `regress` does OLS
  - option `vce(robust)` for heteroskedastic-robust standard errors
  - option `vce(cluster clid)` for cluster-robust standard errors (with cluster on `clid`)
- For Feasible GLS
  - command `regress [aweight= ]` for known or estimated heteroskedasticity
  - command `sureg` for systems of linear equations
  - command `nlsur` for systems of nonlinear equations
  - command `xtreg, re` for panel random effects.
- For hypothesis tests
  - command `test` (and `nltest` for nonlinear hypotheses)

9. Appendix: OLS matrix notation example

- Example: $N = 4$ with (x, y) equal to $(1,1)$, $(2,3)$, $(2,4)$, and $(3,4)$.
- Vector $\mathbf{y}$ and matrix $\mathbf{X}$ are

$$\underset{(4\times 1)}{\mathbf{y}} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 4 \end{bmatrix}$$

and

$$\underset{(4\times 2)}{\mathbf{X}} = \begin{bmatrix} \mathbf{x}_1' \\ \mathbf{x}_2' \\ \mathbf{x}_3' \\ \mathbf{x}_4' \end{bmatrix} = \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \\ x_{13} & x_{23} \\ x_{14} & x_{24} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

- Compute $\widehat{\beta}_{\text{OLS}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$:

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 18 \end{bmatrix}.$$

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} 4 & 8 \\ 8 & 18 \end{bmatrix}^{-1} = \frac{1}{72-64}\begin{bmatrix} 18 & -8 \\ -8 & 4 \end{bmatrix} = \begin{bmatrix} 9/4 & -1 \\ -1 & 1/2 \end{bmatrix}.$$

$$\mathbf{X}'\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 27 \end{bmatrix}.$$

$$(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \begin{bmatrix} 9/4 & -1 \\ -1 & 1/2 \end{bmatrix}\begin{bmatrix} 12 \\ 27 \end{bmatrix} = \begin{bmatrix} 108/4-27 \\ -12+54/4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}.$$

- OLS estimates:
  - intercept $\widehat{\beta}_1 = 0$ and slope coefficient $\widehat{\beta}_2 = 1.5$.

- OLS on intercept and single regressor: $y_i = \beta_1 + \beta_2 x_i + u_i$.
  - $\mathbf{X}'\mathbf{X} = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_N \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_N \end{bmatrix} = \begin{bmatrix} N & \sum_i x_i \\ \sum_i x_i & \sum_i x_i^2 \end{bmatrix}$
  - $(\mathbf{X}'\mathbf{X})^{-1} = \frac{1}{N\sum_i x_i^2 - (\sum_i x_i)^2} \begin{bmatrix} \sum_i x_i^2 & -\sum_i x_i \\ -\sum_i x_i & N \end{bmatrix}$
  $= \frac{1}{\sum_i x_i^2 - N\bar{x}^2} \begin{bmatrix} N^{-1}\sum_i x_i^2 & -\bar{x} \\ -\bar{x} & 1 \end{bmatrix}$
  - $\mathbf{X}'\mathbf{y} = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_N \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} \sum_i y_i \\ \sum_i x_i y_i \end{bmatrix} = \begin{bmatrix} N\bar{y} \\ \sum_i x_i y_i \end{bmatrix}$
  - $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \frac{1}{\sum_i x_i^2 - N\bar{x}^2} \begin{bmatrix} \bar{y}\sum_i x_i^2 - \bar{x}\sum_i x_i y_i \\ -\bar{x}N\bar{y} + \sum_i x_i y_i \end{bmatrix}$
  $= \frac{1}{\sum_i (x_i - \bar{x})^2} \begin{bmatrix} \bar{y}\sum_i x_i^2 - \bar{x}\sum_i x_i y_i \\ \sum_i (x_i - \bar{x})(y_i - \bar{y}) \end{bmatrix} = \begin{bmatrix} \bar{y} - \widehat{\beta}_2\bar{x} \\ \frac{\sum_i (x_i - \bar{x})(y_i - \bar{x})}{\sum_i (x_i - \bar{x})^2} \end{bmatrix}$
- So $\widehat{\beta}_1 = \bar{y} - \widehat{\beta}_2\bar{x}$ and $\widehat{\beta}_2 = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{x})}{\sum_i (x_i - \bar{x})^2}$ as in introductory course.